作业1-13

证明布尔代数是有界有补分配格，有界有补分配格是布尔代数

- 布尔代数是有界有补分配格
  - 给定布尔代数$\langle B, +, *, ', 0, 1\rangle$, 令‘ + ’ = ‘ ∨ ’ ‘ ∗ ’ = ‘ ∧ ’，
    - **格**：由于+，*满足交换律，结合律以及吸收律(Theorem 15.2),所以 给定布尔代数为1个Lattice（Lattice 定义）
    - **有界**：由对于B中任意元素a, 由于$a + 1 = 1, a * 0 = 0$ (Theorem $15.2 - ii$), 可得$a \vee 1 = 1, a \wedge 0 = 0$, 所以$0 \leq a \leq 1$，即给定布尔代数是有界的
    - **有补**：由对于B中任意元素a，都存在a的补元a'(布尔代数定义)， $a + a' = 1, a * a'$=0
    - **分配**：布尔代数运算+，*满足**分配率**(布尔代数定义)
  - 所以布尔代数是有界有补分配格。

证明布尔代数是有界有补分配格，有界有补分配格是布尔代数

- 有界有补分配格是布尔代数
  - 给定有界有补分配格⟨L,∧,∨⟩,令‘ + ’ = ‘ ∨ ’ ‘ ∗ ’ = ‘ ∧ ’
    - 交换律：显然‘ + ’ = ‘ ∨ ’ ‘ ∗ ’ = ‘ ∧ ’满足交换律(格的定义)
    - 分配律：显然‘ + ’ = ‘ ∨ ’ ‘ ∗ ’ = ‘ ∧ ’满足分配率(分配格定义)
    - 同一性：由于有界，必然存在1,0，使得对L中任意元素$0 \le a \le 1$, 所以$a + 0 = a \vee 0 = a, a * 1 = a \wedge 1 = a$
    - 有补律：显然‘ + ’ = ‘ ∨ ’ ‘ ∗ ’ = ‘ ∧ ’满足分配率(有补格定义)
  - 所以，有界有补分配格是布尔代数

Let B be a finite Boolean algebra. Recall (Section 14.10) that an element a in B is an atom if a immediately succeeds 0, that is if $0 \ll a$. Let A be the set of atoms of B and let $P(A)$ be the Boolean algebra of all subsets of the set A of atoms. By Theorem 14.8, each $x \neq 0$ in B can be expressed uniquely (except for order) as the sum (join) of atoms, i.e., elements of A. Say,

$$x = a_1 + a_2 + \cdots + a_r$$

is such a representation. Consider the function $f: B \to P(A)$ defined by

$$f(x) = \{a_1, a_2, \ldots, a_r\}$$

The mapping is well defined since the representation is unique.

Theorem 15.6: The above mapping $f: B \to P(A)$ is an isomorphism.

Two Boolean algebras B and B' are said to be *isomorphic* if there is a one-to-one correspondence $f: B \to B'$ which preserves the three operations, i.e., such that, for any elements, a, b in B,

$$f(a+b) = f(a) + f(b), \quad f(a * b) = f(a) * f(b) \quad \text{and} \quad f(a') = f(a)'$$

1) 易证f是bijective
2) 令$x = a_{x_1} + a_{x_2} + \cdots + a_{x_r}, y = a_{y_1} + a_{y_2} + \cdots a_{y_{r'}}$

$$f(x) = \{a_{x_1}, a_{x_2}, \ldots, a_{x_r}\}, f(y) = \{a_{y_1}, a_{y_2}, \ldots a_{y_{r'}}\}$$

i) $x + y = a_{x_1} + a_{x_2} + \cdots + a_{x_r} + a_{y_1} + a_{y_2} + \cdots a_{y_{r'}} = \sum_{a \in f(x) \cup f(y) - f(x) \cap f(y)} a$ (吸收律)

$f(x+y) = f(\sum_{a \in f(x) \cup f(y) - f(x) \cap f(y)} a) = f(x) \cup f(y) - f(x) \cap f(y)$

$f(x) + f(y) = \{a_{x_1}, a_{x_2}, \ldots, a_{x_r}\} + \{a_{y_1}, a_{y_2}, \ldots, a_{y_{r'}}\} = f(x) \cup f(y) - f(x) \cap f(y)$

$$\therefore f(x+y) = f(x) + f(y)$$

ii) 类似i)

iii) $1 = a_1 + a_2 + \cdots + a_{|B|}$, $f(1) = \{a_1, a_2, \ldots, a_{|B|}\}$

已知$f(x)' = f(1) - f(x)$

$\because x' + x = 1$且a_i均为atom，$\therefore x' = \sum_{a \in f(1) - f(x)} a$

$\therefore f(x') = f(1) - f(x) = f(x)'$

证明等势的（有穷）布尔代数均同构

Two Boolean algebras B and B' are said to be *isomorphic* if there is a one-to-one correspondence $f\colon B \to B'$ which preserves the three operations, i.e., such that, for any elements, a, b in B,

$$f(a+b) = f(a) + f(b), \quad f(a*b) = f(a)*f(b) \quad \text{and} \quad f(a') = f(a)'$$

- 基本思路：
  - 由Theorem 15.6，对任意有穷等式布尔代数X、Y，若其Atom集合为A_X, A_Y 则$X \cong P(A_X), Y \cong P(A_Y)$
  - 已知|X|=|Y|
  - 证明：$\boldsymbol{P(A_X) \cong B^{|X|}} = B^{|Y|} \cong P(A_Y), B^{|X|}$为:

Let $\mathbf{B}^n = \mathbf{B} \times \mathbf{B} \times \cdots \times \mathbf{B}$ (n factors) where the operations of $+$, $*$, and $'$ are defined componentwise using Fig. 15-1. For notational convenience, we write the elements of $\mathbf{B}^n$ as n-bit sequences without commas, e.g., $x = 110011$ and $y = 111000$ belong to $\mathbf{B}^n$. Hence

$$x + y = 111011, \quad x * y = 110000, \quad x' = 001100$$

Then $\mathbf{B}^n$ is a Boolean algebra. Here $0 = 000\cdots0$ is the zero element, and $1 = 111\cdots1$ is the unit element. We note that $\mathbf{B}^n$ has 2^n elements.