3-5 Minimum Spanning Trees

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Uniqueness of MST (Problem 4.30)

$$T \text{ is a unique MST of } G$$
$$\Longleftrightarrow$$
$$\forall e \in G \setminus T : w(e) > w(\text{every other edge on the cycle in } T + e)$$

Uniqueness of MST (Problem 4.29)

$$\text{Distinct weights} \implies \text{Unique MST}$$

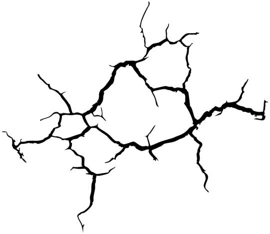
Cut Property

Cycle Property

Be Careful with Your Proofs!!!

Theorem (A "Real" Theorem)

Proof.



□

Theorem (A "Faked" Theorem)

Proof.

□

Cut Property

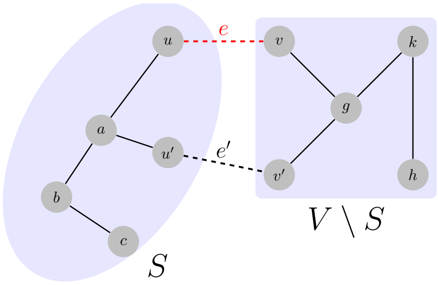
Cut Property (Version I)

$$
\begin{aligned}
X &: \text{A part of some MST } T_1 \text{ of } G \\
(S, V \setminus S) &: \text{A \textit{cut} such that } X \text{ does \textit{not} cross } (S, V \setminus S) \\
e &: \text{A lightest edge across } (S, V \setminus S)
\end{aligned}
$$

Then $X \cup \{e\}$ is a part of some MST T_2 of G.

Correctness of Prim's and Kruskal's algorithms.

By Exchange Argument.



$$T' = \underbrace{\underbrace{T}_{\color{blue}{X \subseteq T}} + \{e\} - \{e'\}}_{\color{red}{\text{if } e \notin T}}$$

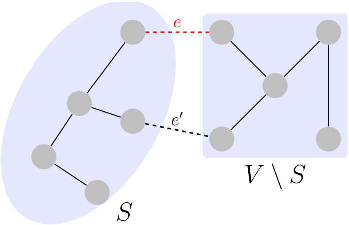
"a" → "the" $\color{red}{\Longrightarrow}$ "some" → "all"

Cut Property (Version II)

A cut $(S, V \setminus S)$

Let $e = (u, v)$ be *a* lightest edge across $(S, V \setminus S)$

$\exists$ MST T of $G : e \in T$



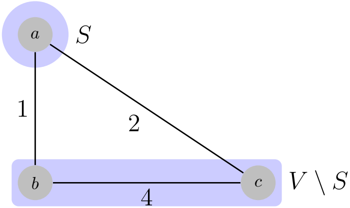
$$T' = \underbrace{T + \{e\}}_{\text{if } e \notin T} - \{e'\}$$

"a" $\to$ "the" $\Longrightarrow$ "$\exists$" $\to$ "$\forall$"

A Wrong Divide&Conquer Algorithm for MST

$$(V_1, V_2) : \Big| |V_1| - |V_2| \Big| \leq 1$$

$$T_1 + T_2 + \{e\} : e \text{ is a lightest edge across } (V_1, V_2)$$

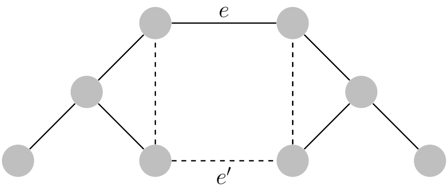


Cycle Property

Cycle Property

- Let C be any cycle in G
- Let $e = (u, v)$ be *a* maximum-weight edge in C

Then $\exists$ MST T of $G : e \notin T$.



$$T' = \underbrace{T - \{e\}}_{\text{if } e \in T} + \{e'\}$$

"a" $\to$ "the" $\implies$ "$\exists$" $\to$ "$\forall$"

Anti-Kruskal Algorithm

Reverse-delete algorithm (wiki; clickable)

$$O\Big(m \log n \; (\log \log n)^3\Big)$$

Proof.

Cycle Property

$$T \subseteq F \implies \exists\, T' : T' \subseteq F - \{e\}$$

□

"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem"

— Kruskal, 1956.

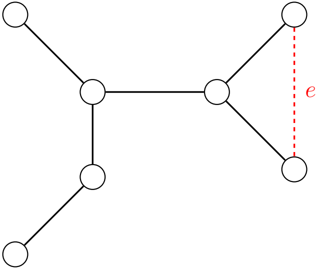
Uniqueness of MST (Problem 4.30)

$$T: \text{ a MST of a connected weight graph } G$$

$$T \text{ is a unique MST of } G$$

$$\iff$$

$$\forall e \in G \setminus T : w(e) > w(\text{every other edge on the cycle in } T + e)$$



Uniqueness of MST

Uniqueness of MST (Problem 4.29)

Distinct weights $\implies$ Unique MST

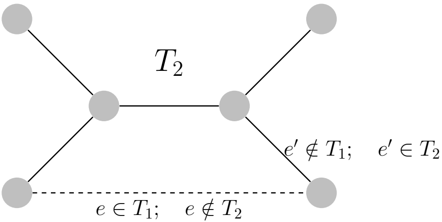
By Contradiction.

$$\exists \text{ MSTs } T_1 \neq T_2$$

$$\Delta E = \{e \mid e \in T_1 \setminus T_2 \lor e \in T_2 \setminus T_1\}$$

$$e = \min \Delta E$$

$$e \in T_1 \setminus T_2 \ \textit{(w.l.o.g)}$$



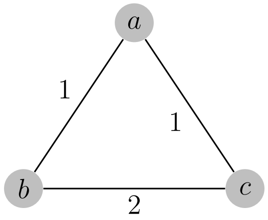
$$T_2 + \{e\} \implies C$$

$$\exists (e' \in C) \notin T_1 \implies e' \in T_2 \setminus T_1 \implies e' \in \Delta E \implies w(e') > w(e)$$

$$T' = T_2 + \{e\} - \{e'\} \implies w(T') < w(T_2)$$

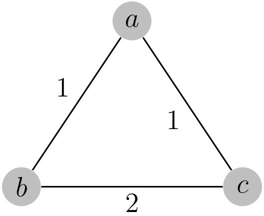
Condition for Uniqueness of MST

$$\text{Unique MST} \;\;\not\Longrightarrow\;\; \text{Distinct weights}$$



Unique MST

Unique MST $\not\Longrightarrow$ Minimum-weight edge across any cut is unique

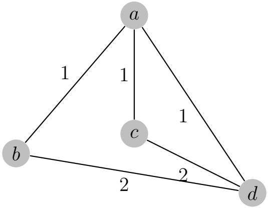


Theorem (After-class Exercise)

Minimum-weight edge across any cut is unique $\Longrightarrow$ *Unique MST*

Unique MST

Unique MST $\not\Rightarrow$ Maximum-weight edge in any cycle is unique



Theorem (After-class Exercise)

Maximum-weight edge in any cycle is unique $\Longrightarrow$ *Unique MST*

Existence of Cycle (Problem 4.8)

$$\forall v \in V(G) : \deg(v) \geq 2 \implies G \text{ contains a cycle}$$

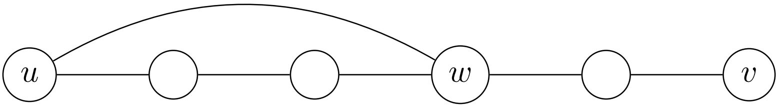
By Contradiction.

$$m = n - k(G) \leq n - 1$$

$$\sum_{v \in V(G)} \deg(v) \leq 2(n - 1)$$

Existence of Cycle (Problem 4.8)

$$\forall v \in V(G) : \deg(v) \geq 2 \implies G \text{ contains a cycle}$$



maximal path $P_{u,v}$

Existence of Cycle (Problem 4.8)

$$\forall v \in V(G) : \deg(v) \geq 2 \implies G \text{ contains a cycle}$$

$$\forall v \in V(G) : \deg(v) \geq 2$$

$$\implies \nexists v \in V(G) : \deg(v) = 1$$

$$\xRightarrow{\text{Theorem 4.3}} G \text{ is not a tree}$$

Consider each component G' of G

$$\xRightarrow{\text{Theorem 4.2}} \exists u, v \in V(G') : u, v \text{ are connected by } \geq 2 \text{ paths}$$

$$\implies G' \text{ contains a cycle}$$

$$\implies G \text{ contains a cycle}$$

Bridge and Spanning Trees (Problem 4.26)

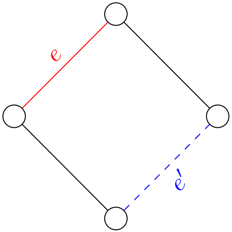
$$G: \text{ a connected graph}$$

$$e \in E(G) \text{ is a bridge} \iff e \in \forall \text{ST of } G$$

" $\Longleftarrow$ "

By Contradiction.

ST of $G - e$

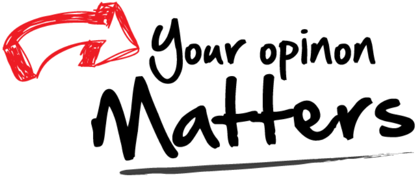


$$T' = \underbrace{T - \{e\}}_{e \in T} + \{e'\}$$

MST from the point of view of greedy-algorithm

MIT 6.046J: *"Design and Analysis of Algorithms"*, Spring 2015


THANK YOU



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