作业反馈3-10

CZ 5.8

CZ 5.10

CZ 5.22

5.8(a) Let G be a nontrivial connected graph. Prove that if v is an end-vertex of a spanning tree of G, then v is not a cut vertex of G.

Proof: [Contrapositive] Suppose that v is a cut vertex of G. Then Corollary 5.4 implies that there are vertices u and w in V(G) distinct from v and each other such that each u - w path in G contains v. Let T be any spanning tree in G. We shall show that v is not an end-vertex of T. To see this, observe that Theorem 4.2 implies that there is a unique u - w path in T. This is also a u - w path in G, and thus must contain v. Thus, it follows that we must have $\deg_T(v) \geq 2$. Thus v is not an end-vertex of T. Since T was an arbitrary spanning tree of G, v will not be an end-vertex of any spanning tree of G.//

5.8(b) Use (a) to give an alternative proof of the fact that every nontrivial connected graph contains at least two vertices that are not cut-vertices.

Proof: [Direct] By Theorem 4.10, G must have at least one spanning tree T. Theorem 4.3 implies that T must have at least two end-vertices. From Part (a), each of these must fail to be a cut-vertex of G.//

5.8 (c) Let v be a vertex in a nontrivial connected graph G. Show that there exists a spanning tree of G that contains all edges of G that are incident with v.

Proof A BFS tree rooting v contains all edges incident to v.

Proof: Observe that the tree T_0 consisting of v, together with all of the neighbors of v and the edges incident with v, is a subgraph of G. There is a maximal tree T in G containing T_0 as a subgraph.

We claim that T must be a spanning tree. Suppose not. Then there is at least one vertex w in G but not in T such that

$$d(w,T) = \min\{ d(w,u): u \in V(T) \}$$

is smallest amongst vertices w not T.

We claim that d(w,T) = 1. Suppose not. Then there is some u in T with d(u,w) = d(w,T) = k > 1. Let P: $u = v_0, v_1, \ldots, v_k = w$ be a u – w geodesic in G. If v_1 is in V(T), then u is not closest to w. On the other hand, if v_1 is not in V(T), then w doesn't give the smallest value of d(w.T) amongst vertices of G not in V(T). Thus, it must follow that d(w,T) = 1.

Now this allows us to contradict the presumed maximality of T, for the tree $T_1 = (V(T) \cup \{w\}, E(T) \cup \{ uw \})$, where $u \in V(T)$ satisfies d(u,w) = d(w,T) = 1, is a tree containing T_0 and properly containing T. Thus, T must in fact span G.//

5.8 (d) Prove that if a connected graph G has exactly two vertices that are not cut-vertices, then G is a path. [Recall that if a tree contains a vertex of degree exceeding 2, then T has more than 2 end-vertices.]

Proof If the graph is not a path, it has a spanning tree with a vertex of degree at least 3. If a tree contains a vertex of degree at least 3, it has at least 3 end-vertices. According to (a), there are not cut-vertices, contradicting the assumption that the graph contains only 2 vertices that are not cut-vertices.

首先,如果图只有两个顶点,结论显然成立.

对于顶点数大于2的情况

1. G无环,则G是一颗树,由(a),G只能有两个端点,因此G是一条路
2. G有环,考察G的所有生成树T,由(a),T都只能有两个端点,即T是一条路,这说明G的所有顶点都在环上,这和G恰有两个非割点矛盾,因此G不能有环

因此G是一条路.

5.10 Prove that a connected graph G of size at least 2 is non-separable if and only if any two adjacent edges of G lie on a common cycle of G

- G is non-separable →any two adjacent edges of G lie on a common cycle
  - If |G.V|=2, obvious
  - |G.V|>=3
    - Let *uv* , *vw* be any pair of two adjacent edges of G
    - u, w must lie on a common cycle C1 of G.
      - There must be a *u*-*w* path P does not contain *v*
      - *P+uv+vw* forms a cycle of G

5.10 Prove that a connected graph G of size at least 2 is non-separable if and only if any two adjacent edges of G lie on a common cycle of G

- any two adjacent edges of G lie on a common cycle →G is non-separable
  - Assume that G is separable and *v* is a cut-vertex of G
  - *v* must be adjacent to at least 2 edges, say *uv, vw;*
  - *uv, vw lie on a common cycle of G*→ there is a u-w path P, which does not contain v
  - So, v cannot be an cut-vertex

5.22(a)Prove that if *G* is a k-connected graph and *e* is an edge of *G*, then *G-e* is (k-1)-connected

(a) *Proof.* 假设 e 的一个端点是 v. 则, $G-v$ 是 $k-1$ 连通的. 而 $G-v$ 是 $G-e$ 的子图. 所以 $G-e$ 是 $k-1$ 连通的. □

(a)If G is a k-connected graph, then K(G) ≧ k. Assume the graph we obtain after removing k-1 vertices of G is called G'. If e is a bridge of G', then G'-e is disconnected and K(G-e) ≧ k-1. Else, G-e is connected and K(G-e) ≧ k≧ k-1. So (G-e) is k-1 connected.

5.22(a)Prove that if *G* is a k-connected graph and *e* is an edge of *G*, then *G-e* is (k-1)-connected

a. G is k-connected,G' = G - e, thus $\lambda(G) \geq \kappa(G) \geq k$.

Case I: $e \in$ U denoting minimal cut edge vertices of G, then D denotes minimal cut edge of G-e, |D|=|U|-1, $\kappa(G') \leq \lambda(G') = \lambda(G) - 1$. if $\lambda(G) = \kappa(G) = k$, then $\kappa(G') = \kappa(G) - 1 = k - 1$, G' is (k-1)-connected.

Case II: otherwise, $e \notin U$ or something else, G is still k-connected absolutely (k-1)-connected.

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5.22(a)Prove that if *G* is a k-connected graph and *e* is an edge of *G*, then *G-e* is (k-1)-connected

a :由 G 为 K 连通图知：$K \leqslant \kappa(G) \leqslant \lambda(G)$

令 A 为 G 的最小点割集，B 为 G'的最小点割集，则 e 存在三种情况。

1：e 为最小点割集中任意两点间的边。

2：e 为最小点割集中的顶点和最小点割集之外顶点间的边。

3：e 为最小点割集之外的任意两个顶点间的边。

对于情况 1，3：知 $\lambda(G') = \lambda(G)$,故而 G'为 K 连通的必为 K-1 连通

对于情况 2：则 G'恰为 K-1 连通。

5.22(a)Prove that if *G* is a k-connected graph and *e* is an edge of *G*, then *G-e* is (k-1)-connected

(a) G is k-connected, so there is a minimum vertex-cut of size k. After removing e from G, at most one vertex is no longer needed in the minimum vertex-cut. Thus G is (k - 1)-connected.

a

若不然，至少存在一个势为$k-2$的顶点集K,使得$G-K-e$不连通,考察e的位置,若e与K中顶点相连,则$G-K$本身不连通,这与G为k连通图矛盾,若e与K中顶点不相连,由于$G-K$是联通的,这说明e是$G-K$的割边,即G是$k-1$连通图,这与G为k连通图矛盾,综上,假设不成立,$G-e$是$k-1$连通图.

(b)Prove that if G is a k-edge-connected graph and e is an edge of G, then G-e is (k-1)-edge-connected

- Case 1: e belongs to an minimum edge-cut set
- Case 2: e does not belong to any minimum edge-cut set