

问题讨论

2013-12-27

Problem 15.1.

Find the compositions $f \circ g$ and $g \circ f$ assuming the domain of each is the largest set of real numbers for which the functions f , g , $f \circ g$, and $g \circ f$ make sense. In your solution to each of the following, give the compositions and the corresponding domain and range:

- (a) $f(x) = 1/(1 + x)$, $g(x) = x^2$;
- (b) $f(x) = x^2$, $g(x) = \sqrt{x}$ (simplify this one);
- (c) $f(x) = 1/x$, $g(x) = x^2 + 1$;
- (d) $f(x) = |x|$, $g(x) = f(x)$.

Problem 15.14.

Let A, B, C , and D be nonempty sets. Let $f : A \rightarrow B$ and $g : C \rightarrow D$ be functions.

- (a) Prove that if f and g are one-to-one, then $H : A \times C \rightarrow B \times D$ defined by

$$H(a, c) = (f(a), g(c))$$

is a one-to-one function. (Check that it is one-to-one and a function.)

- (b) Prove that if f and g are onto, then H is also onto.

To say that a function $f : A \rightarrow B$ is **one-to-one** means that for all $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$. A function $f : A \rightarrow B$ is **onto** if $\text{ran}(f) = B$. If a function has this property, then we say

Problem 15.20.

In this problem, we look at a function called the restriction function, which we now define.

If $f : A \rightarrow B$ is a function, and $A_1 \subset A$, we define another function $F : A_1 \rightarrow B$ by $F(a) = f(a)$ for all $a \in A_1$. This function F is called the **restriction** of f to A_1 and is usually denoted $f|_{A_1}$. We now turn to the problem:

- (a) Prove that if f is one-to-one, then $f|_{A_1}$ is one-to-one.
- (b) Prove that if $f|_{A_1}$ is onto, then f is onto.

Problem 15.12.

Let $f : A \rightarrow A$ be a function. Define a relation on A by $a \sim b$ if and only if $f(a) = f(b)$. Is this an equivalence relation? If f is one-to-one, what is the equivalence class of a point $a \in A$?

Problem 15.13.

Let $f : A \rightarrow A$ be a function. Define a relation on A by $a \sim b$ if and only if $f(a) = b$. Is this an equivalence relation for an arbitrary function f ? If not, is there a function for which it is an equivalence relation?

Once we have an equivalence relation on a set X , we define the **equivalence class** of an element $x \in X$ to be the set E_x where $E_x = \{y \in X : x \sim y\}$. Using this notation, we see that the sets

Problem 16.20.

Suppose that $f : X \rightarrow Y$ is a function, and let A_1 and A_2 be subsets of X .

- (a) If $f(A_1) = f(A_2)$, must $A_1 = A_2$?
- (b) Let f be a bijective function. Show that if $f(A_1) = f(A_2)$, then $A_1 = A_2$. Indicate clearly where you use one-to-one or onto. Did you use both?

Problem 20.9.

- (a) Suppose that A and B are nonempty finite sets and $A \cap B = \emptyset$. Show that there exist integers n and m such that $A \approx \{1, 2, \dots, n\}$ and $B \approx \{n + 1, \dots, n + m\}$.
- (b) Prove Corollary 20.8.

allows us to give a precise definition of a finite set. We say that a set S is **finite** if either $S = \emptyset$ or if S is equivalent to the set $\{1, 2, 3, \dots, n\}$ for some positive integer n . Thus, to prove that a nonempty set is

Problem 21.11.

Let X be an infinite set, and A and B be finite subsets of X . Answer each of the following, giving reasons for your answers:

- (a) Is $A \cap B$ finite or infinite?
- (b) Is $A \setminus B$ finite or infinite?
- (c) Is $X \setminus A$ finite or infinite?
- (d) Is $A \cup B$ finite or infinite?
- (e) If $f : A \rightarrow X$ is a one-to-one function, is $f(A)$ finite or infinite?

Problem 21.19.

Let A be a finite set. Show that a function $f : A \rightarrow A$ is one-to-one if and only if it is onto. Is this still true if A is infinite?

Problem 22.1.

Give an example, if possible, of each of the following:

- (a) a countably infinite collection of pairwise disjoint finite sets whose union is countably infinite; (See Problem 8.11 for the definition of pairwise disjoint.)
- (b) a countably infinite collection of nonempty sets whose union is finite;
- (c) a countably infinite collection of pairwise disjoint nonempty sets whose union is finite.

Problem 22.2.

Which of the following sets are finite? countably infinite? uncountable? (Be careful—don't apply theorems for finite sets to infinite sets!) Give reasons for your answers for each of the following:

- (a) $\{1/n : n \in \mathbb{Z} \setminus \{0\}\};$
- (b) $\mathbb{R} \setminus \mathbb{N};$
- (c) $\{x \in \mathbb{Z} : |x - 7| < |x|\};$
- (d) $2\mathbb{Z} \times 3\mathbb{Z};$
- (e) the set of all lines with rational slopes;
- (f) $\mathbb{Q} \setminus \{0\};$
- (g) $\mathbb{N} \setminus \{1, 3\}.$

Problem 22.3.

Is the set of all infinite sequences of 0's and 1's finite, countably infinite, or uncountable? Guess and then prove, please.

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