Open Topic: homework CZ: 9.8,10.5

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CZ:9.8

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CZ:9.8

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2. planar graph

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2. $\times$: Cartesian product

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2. $\times$: Cartesian product

Cartesian product

For two graphs G and H, the **Cartesian product** $G \times H$ has vertex set $V(G \times H) = V(G) \times V(H)$, that is, every vertex of $G \times H$ is an order pair (u, v), where $u \in V(G)$ and $v \in V(H)$. Two distinct vertices (u,v) and (x,y) are adjacent in $G \times H$ if either
(1) $u = x$ and $vy \in E(H)$ or
(2) $v = y$ and $ux \in E(G)$.

the graph $K_4 \times K_2$

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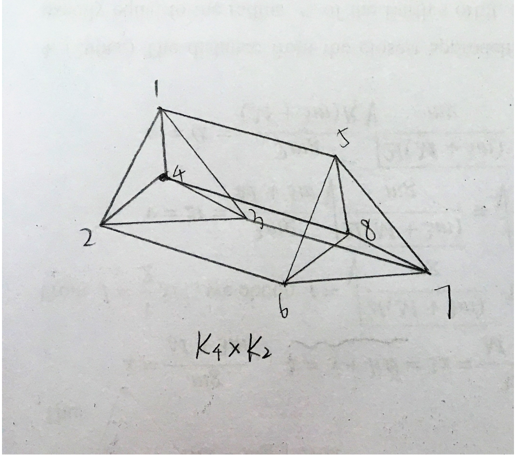
emmmmm, the definition is too long! TAT

$K_4 \times K_2$

手画的图，有点丑，见谅



planar graph

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How to determine whether a graph is a planar graph or not?

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1. Theroem 9.2 (It may not works for all graphs.)

If G is a planar graph of order $n \geq 3$ and size m, then $m \leq 3n - 6$.

> "It provides a necessary condition for a graph to be planar and so provides a sufficient condition for a graph to be nonplanar."(CZ)

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> $m = 6 \times 2 + 4 = 16, n = 4 \times 2 = 8, 3n - 6 = 18, m < 3n - 6$

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Can't get the answer.

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2. Theroem 9.7 (Kuratowski's Theorem)

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A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.

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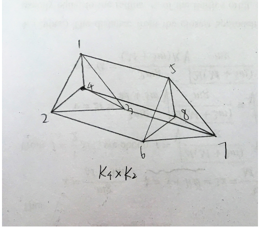
- subdivision

A graph G' is called a **subdivision** of a graph G if $G' = G$ or one or more vertices of degree 2 are inserted into one or more edges of G.

Kuratowski's Theorem



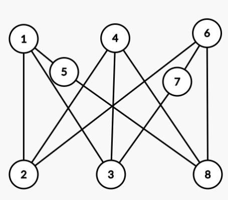


图: a subdivision of $K_{3,3}$

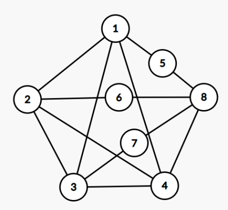


图: a subdivision of K_5

CZ:9.8

Question9.8

Determine, with explanation, whether the graph $K_4 \times K_2$ is planar.

Proof.

For the graph $K_4 \times K_2$ has the subdivision of K_5 (or $K_{3,3}$), so it is not planar. □

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Question10.5

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Definition

By a **proper coloring** (or, more simply, a **coloring**) of a graph G, we mean an assignmenrt of colors (elements of some set) to the vertices of G, one color to each vertex, such that adjacent vertices are colored differently.
The smallest number of colors in any coloring of a graph G is called the **chromatic number** of G and is denoted by $\chi(G)$.

Mentality

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1. color classes （色类）
2. clique number （团数）

Considering the color classes - Proof 1

About color classes

If G is a k-chromatic graph, then it is possible to partition V(G) into k independent sets $V_1, V_2, \cdots, V_k$, called **color classes**.

- Each color class is an independent sets.
- As described in the question, consider A is graph of order 6 with chromatic number 3.

Considering the color classes - Proof 1

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- So A is 3-chromatic.

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- Because A is of order 6, the number of vertices in each color classes of a 3-coloring of A has 3 situations as follows

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1. 2,2,2
2. 3,2,1
3. 4,1,1

Considering the color classes - Proof 1

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- We konw that K_6 has 15 edges. Suppose that A has m edges. Consider the 3 situations described above.

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Proof 2 ended.

Considering the clique number - Proof 3

clique number

A **clique** in a graph G is a complete subgraph of G. The order of the largest clique in a graph G is its **clique number**, which is denoted by $\omega(G)$.

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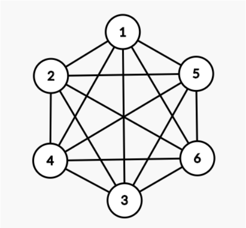
And we have Theorem 10.5:

Theorem 10.5

For every graph G of order n, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{n}{\alpha(G)}$.

Considering the clique number - Proof 3



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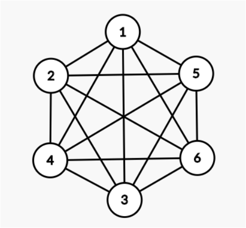

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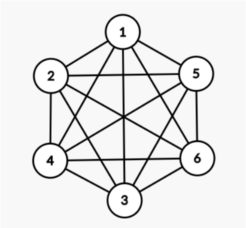

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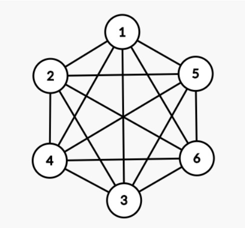

- Considering to construct A by deleting edges in K_6
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- When delete two edges, the $\omega(A)$ is at least 4;

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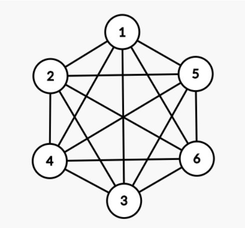

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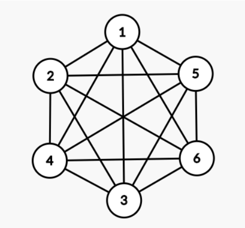

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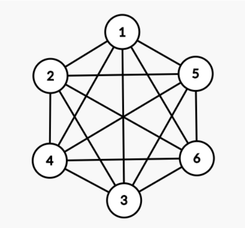

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Considering the clique number - Proof 4

Turàn's theorem

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Let G be any graph with n vertices, such that G is K_{r+1}-free. Then the number of edges in G is at most

$$\frac{r-1}{r} \cdot \frac{n^2}{2} = (1 - \frac{1}{r}) \cdot \frac{n^2}{2}.$$

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- We know that $\omega(A) \leq \chi(A) = 3$, so A is K_4-free. So the number of edges in G is at most $\frac{2}{3} \cdot \frac{36}{2} = 12$.

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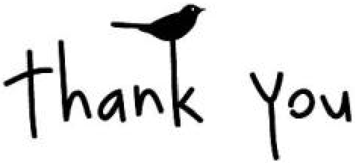
thank You