1-8 Set Theory: Axioms and Operations

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Set Operations (I)

$\cap \qquad \cup \qquad \setminus$

UD Problem 7.1 (d)

Let $A, B \subseteq X$.

$$A \subseteq B \iff (X \setminus B) \subseteq (X \setminus A)$$

$$Q : A, B \subseteq X?$$

1. $A \subseteq B \implies (X \setminus B) \subseteq (X \setminus A)$

   By Contradiction.

2. $(X \setminus B) \subseteq (X \setminus A) \implies A \subseteq B$

   By Contradiction.

(2) needs $A \subseteq X$

UD Problem 7.1 (f)

$$A \cap B = B \iff B \subseteq A$$

UD Problem 7.2

Let $A, B \subseteq X$.

$$A \cap B = \emptyset \iff B \subseteq (X \setminus A)$$

$Q : A, B \subseteq X?$

We need only $B \subseteq X$.

UD Problem 7.19

Let $A, B, C \subseteq X$.

$$A \cap (B^c \cap C^c) = \emptyset \iff A \subseteq B \cup C$$

UD Problem 7.14

Let $A, B \subseteq X$. Prove that the union of two sets can be rewritten as the union of two disjoint sets.

(a) Prove that $(A \setminus B) \cap B = \emptyset$

(b) Prove that $A \cup B = (A \setminus B) \cup B$

By Contradiction.

$(A \setminus B) \cup B = \dots$

$(A \cap \bar{B}^{(X)}) \cup B$

“太容易了，一时没反应过来”

$A, B \subseteq X$ is not necessary.

UD Problem 7.20

$$(A \cup B) \setminus (C \cup D) = (A \setminus (C \cup D)) \cup (B \setminus (C \cup D))$$

$$E \triangleq C \cup D$$

Set Operations (II)

$\cap \quad \cup$

UD Problem 8.1

$$A_n = [0, 1/n) \quad B_n = [0, 1/n] \quad C_n = (0, 1/n)$$

(a) $\bigcup_{n=1}^{\infty} A_n = [0, 1) \quad \bigcup_{n=1}^{\infty} B_n = [0, 1] \quad \bigcup_{n=1}^{\infty} C_n = (0, 1)$

UD Problem 8.1

$$A_n = [0, 1/n) \quad B_n = [0, 1/n] \quad C_n = (0, 1/n)$$

(b) $\bigcap_{n=1}^{\infty} A_n = \{0\}$ $\quad \bigcap_{n=1}^{\infty} B_n = \{0\}$ $\quad \bigcap_{n=1}^{\infty} C_n = \emptyset$

Proof.

□

UD Problem 8.1

$$A_n = [0, 1/n) \quad B_n = [0, 1/n] \quad C_n = (0, 1/n)$$

(b) $\bigcap_{n=1}^{\infty} A_n = \{0\} \quad \bigcap_{n=1}^{\infty} B_n = \{0\} \quad \bigcap_{n=1}^{\infty} C_n = \emptyset$

Theorem (The Nested Interval Theorem (Cantor))

设 $\{[a_n, b_n]\}$ 为递降闭区间套序列，即

$$[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots.$$

如果 $\lim_{n \to \infty} (b_n - a_n) = 0$，则存在唯一的点 c，使得 $c \in [a_n, b_n], \forall n \geq 1$.

$$\forall n \in \mathbb{Z}^+ : A_n \subset B_n \not\Rightarrow \bigcap_{n=1}^{\infty} A_n \subset \bigcap_{n=1}^{\infty} B_n$$

UD Problem 8.14

$$A = \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\})$$

$$X_n = \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$$

$$\begin{aligned} A &= \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} \setminus X_n) \\ &= \mathbb{R} \setminus \Big(\mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}^+} X_n\Big) \\ &= \mathbb{R} \setminus \Big(\mathbb{R} \setminus \mathbb{Z}\Big) \\ &= \mathbb{Z} \end{aligned}$$

UD Problem 8.15

$$A = \mathbb{Q} \setminus \bigcap_{n \in \mathbb{Z}} (\mathbb{R} \setminus \{2n\})$$

Q : What is the temporary universe?

$$\begin{aligned} A &= \mathbb{Q} \setminus \bigcap_{n \in \mathbb{Z}} (\mathbb{R} \setminus \{2n\}) \\ &= \mathbb{Q} \setminus \Big(\mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}} \{2n\}\Big) \\ &= \mathbb{Q} \setminus \Big(\bigcup_{n \in \mathbb{Z}} \{2n\}\Big)^c \\ &= \mathbb{Q} \cap \bigcup_{n \in \mathbb{Z}} \{2n\} \\ &= \{2n : n \in \mathbb{Z}\} \end{aligned}$$

Set Operations (III)

$$\mathcal{P}(X)$$

$$\boxed{S \in \mathcal{P}(X) \iff S \subseteq X}$$

UD Problem 9.8

$$A \subseteq B \iff \mathcal{P}(A) \subseteq \mathcal{P}(B)$$

$$A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B)$$

$$\begin{aligned} & x \in \mathcal{P}(A) \\ \implies & x \subseteq A \\ \implies & x \subseteq B \\ \implies & x \in \mathcal{P}(B) \end{aligned}$$

$$\mathcal{P}(A) \subseteq \mathcal{P}(B) \implies A \subseteq B$$

$$\begin{aligned} & x \in A \\ \implies & \{x\} \subseteq A \\ \implies & \{x\} \in \mathcal{P}(A) \\ \implies & \{x\} \in \mathcal{P}(B) \\ \implies & \{x\} \subseteq B \\ \implies & x \in B \end{aligned}$$

UD Problem 9.9

$$\bigcup_{\alpha \in I} \mathcal{P}(A_\alpha) \subseteq \mathcal{P}(\bigcup_{\alpha \in I} A_\alpha)$$

$$\begin{aligned} & x \in \bigcup_{\alpha \in I} \mathcal{P}(A_\alpha) \\ \implies & \exists \alpha \in I : x \in \mathcal{P}(A_\alpha) \\ \implies & \exists \alpha \in I : x \subseteq A_\alpha \\ \implies & x \subseteq \bigcup_{\alpha \in I} A_\alpha \\ \implies & x \in \mathcal{P}(\bigcup_{\alpha \in I} A_\alpha) \end{aligned}$$

UD Problem 9.10

$$\bigcap_{\alpha \in I} \mathcal{P}(A_\alpha) = \mathcal{P}(\bigcap_{\alpha \in I} A_\alpha)$$

$$\begin{aligned}
& x \in \bigcap_{\alpha \in I} \mathcal{P}(A_\alpha) \\
\iff & \forall \alpha \in I : x \in \mathcal{P}(A_\alpha) \\
\iff & \forall \alpha \in I : x \subseteq A_\alpha \\
\iff & x \subseteq \bigcap_{\alpha \in I} A_\alpha \\
\iff & x \in \mathcal{P}(\bigcap_{\alpha \in I} A_\alpha)
\end{aligned}$$

UD Problem 9.19

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$$

$$(a, d) \in A \times (B \setminus C)$$


Thank
You!