计算机问题求解 – 论题4-17

- 去随机化

课程研讨

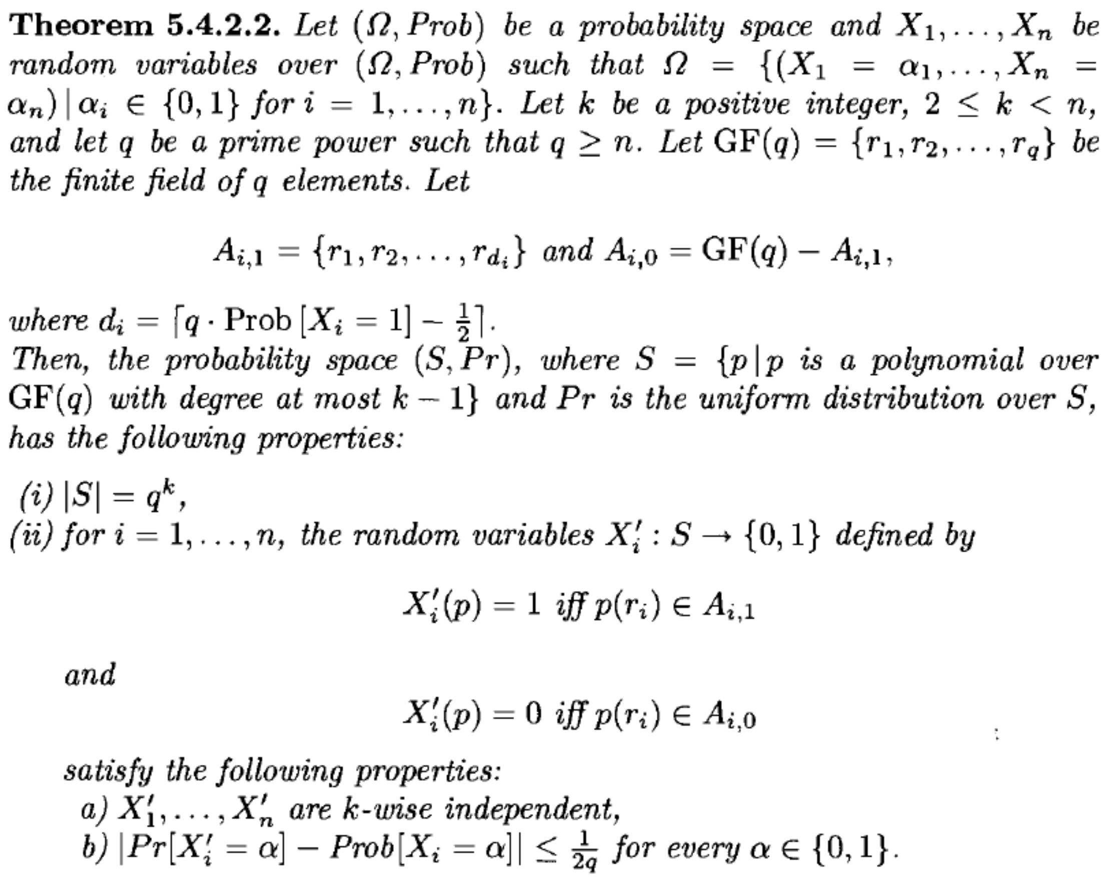
- JH第5章第4节

问题1：去随机

- 去随机，在算法设计上追求的目标是什么？
  - To obtain a required approximation ratio or correct results efficiently for all input instances rather than merely with a high probability.
  - Without any essential increase in the amount of computational resources.

问题2：reduction of the probability space size

- 这种方法的基本思路是什么？
- 定理5.4.2.2是如何具体实现这一思路的？



Theorem 5.4.2.2. *Let $(\Omega, Prob)$ be a probability space and $X_1, \ldots, X_n$ be random variables over $(\Omega, Prob)$ such that $\Omega = \{(X_1 = \alpha_1, \ldots, X_n = \alpha_n) \,|\, \alpha_i \in \{0,1\}$ for $i = 1, \ldots, n\}$. Let k be a positive integer, $2 \leq k < n$, and let q be a prime power such that $q \geq n$. Let $\mathrm{GF}(q) = \{r_1, r_2, \ldots, r_q\}$ be the finite field of q elements. Let*

$$A_{i,1} = \{r_1, r_2, \ldots, r_{d_i}\} \text{ and } A_{i,0} = \mathrm{GF}(q) - A_{i,1},$$

where $d_i = \lceil q \cdot \mathrm{Prob}\,[X_i = 1] - \frac{1}{2} \rceil$.
Then, the probability space (S, Pr), where $S = \{p \,|\, p$ is a polynomial over $\mathrm{GF}(q)$ with degree at most $k-1\}$ and Pr is the uniform distribution over S, has the following properties:

(i) $|S| = q^k$,
(ii) for $i = 1, \ldots, n$, *the random variables* $X_i' : S \to \{0,1\}$ *defined by*

$$X_i'(p) = 1 \text{ iff } p(r_i) \in A_{i,1}$$

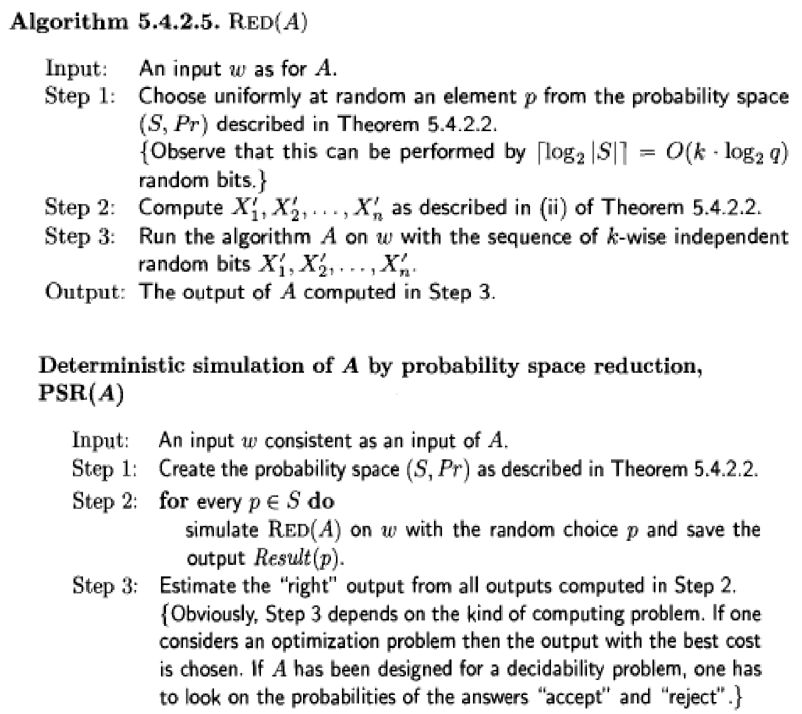
and

$$X_i'(p) = 0 \text{ iff } p(r_i) \in A_{i,0}$$

satisfy the following properties:
a) $X_1', \ldots, X_n'$ *are k-wise independent,*
b) $|Pr[X_i' = \alpha] - Prob[X_i = \alpha]| \leq \frac{1}{2q}$ *for every* $\alpha \in \{0,1\}$.

问题2：reduction of the probability space size (续)

- 如何利用定理5.4.2.2来减少一个随机算法的random bits并最终去随机？
- 对于MAX-E*k*SAT是如何具体实现的？
- 你如何评价这种方法的理论和实际意义？



Algorithm 5.4.2.5. RED(A)

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Input:   An input $w$ as for $A$.
Step 1:  Choose uniformly at random an element $p$ from the probability space
         $(S, Pr)$ described in Theorem 5.4.2.2.
         {Observe that this can be performed by $\lceil \log_2 |S| \rceil = O(k \cdot \log_2 q)$
         random bits.}
Step 2:  Compute $X_1', X_2', \ldots, X_n'$ as described in (ii) of Theorem 5.4.2.2.
Step 3:  Run the algorithm $A$ on $w$ with the sequence of $k$-wise independent
         random bits $X_1', X_2', \ldots, X_n'$.
Output:  The output of $A$ computed in Step 3.
```

Deterministic simulation of A by probability space reduction, PSR(A)

```
Input:   An input $w$ consistent as an input of $A$.
Step 1:  Create the probability space $(S, Pr)$ as described in Theorem 5.4.2.2.
Step 2:  for every $p \in S$ do
           simulate RED($A$) on $w$ with the random choice $p$ and save the
           output Result($p$).
Step 3:  Estimate the "right" output from all outputs computed in Step 2.
         {Obviously, Step 3 depends on the kind of computing problem. If one
         considers an optimization problem then the output with the best cost
         is chosen. If $A$ has been designed for a decidability problem, one has
         to look on the probabilities of the answers "accept" and "reject".}
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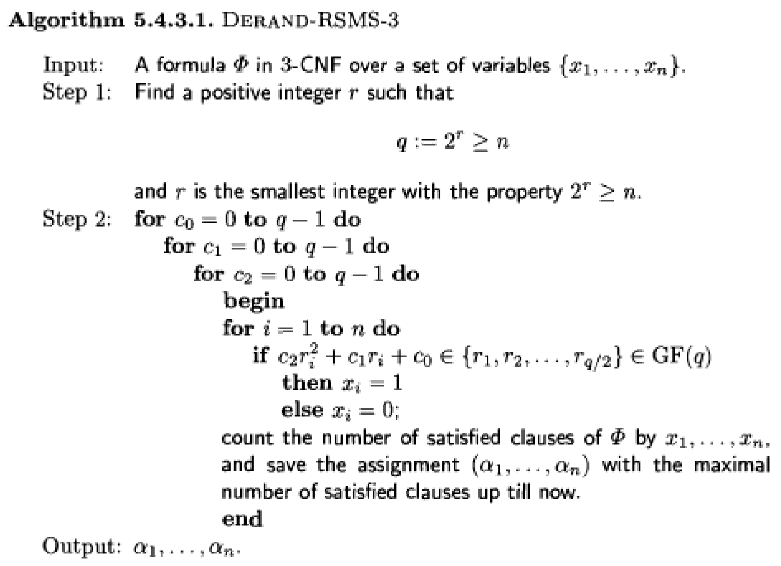


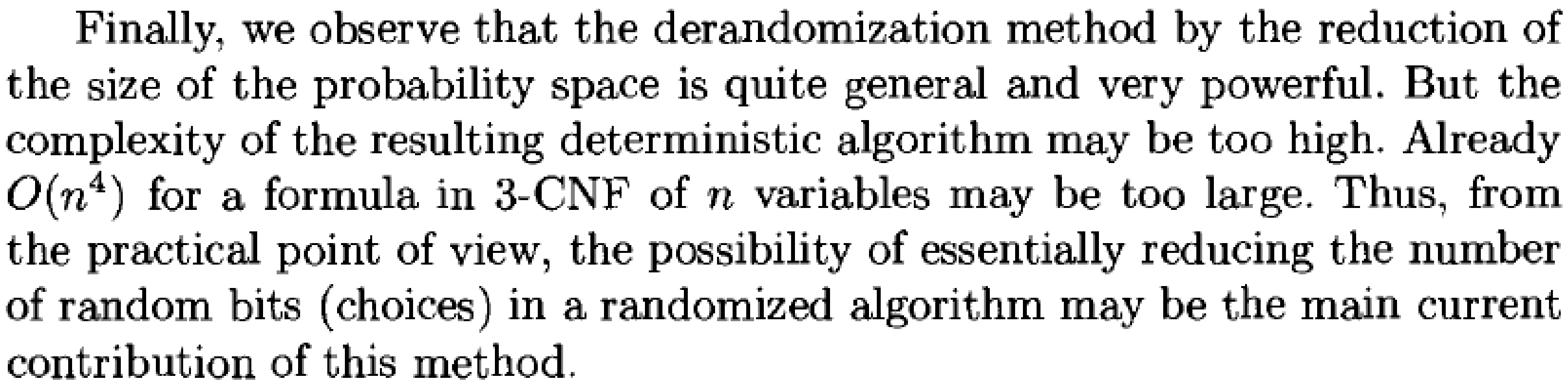
Algorithm 5.4.3.1. DERAND-RSMS-3

```
Input:   A formula $\Phi$ in 3-CNF over a set of variables $\{x_1, \ldots, x_n\}$.
Step 1:  Find a positive integer $r$ such that

                   $q := 2^r \geq n$

         and $r$ is the smallest integer with the property $2^r \geq n$.
Step 2:  for $c_0 = 0$ to $q-1$ do
           for $c_1 = 0$ to $q-1$ do
             for $c_2 = 0$ to $q-1$ do
               begin
               for $i = 1$ to $n$ do
                 if $c_2 r_i^2 + c_1 r_i + c_0 \in \{r_1, r_2, \ldots, r_{q/2}\} \in GF(q)$
                   then $x_i = 1$
                   else $x_i = 0$;
               count the number of satisfied clauses of $\Phi$ by $x_1, \ldots, x_n$,
               and save the assignment $(\alpha_1, \ldots, \alpha_n)$ with the maximal
               number of satisfied clauses up till now.
               end
Output:  $\alpha_1, \ldots, \alpha_n$.
```

问题2：reduction of the probability space size (续)

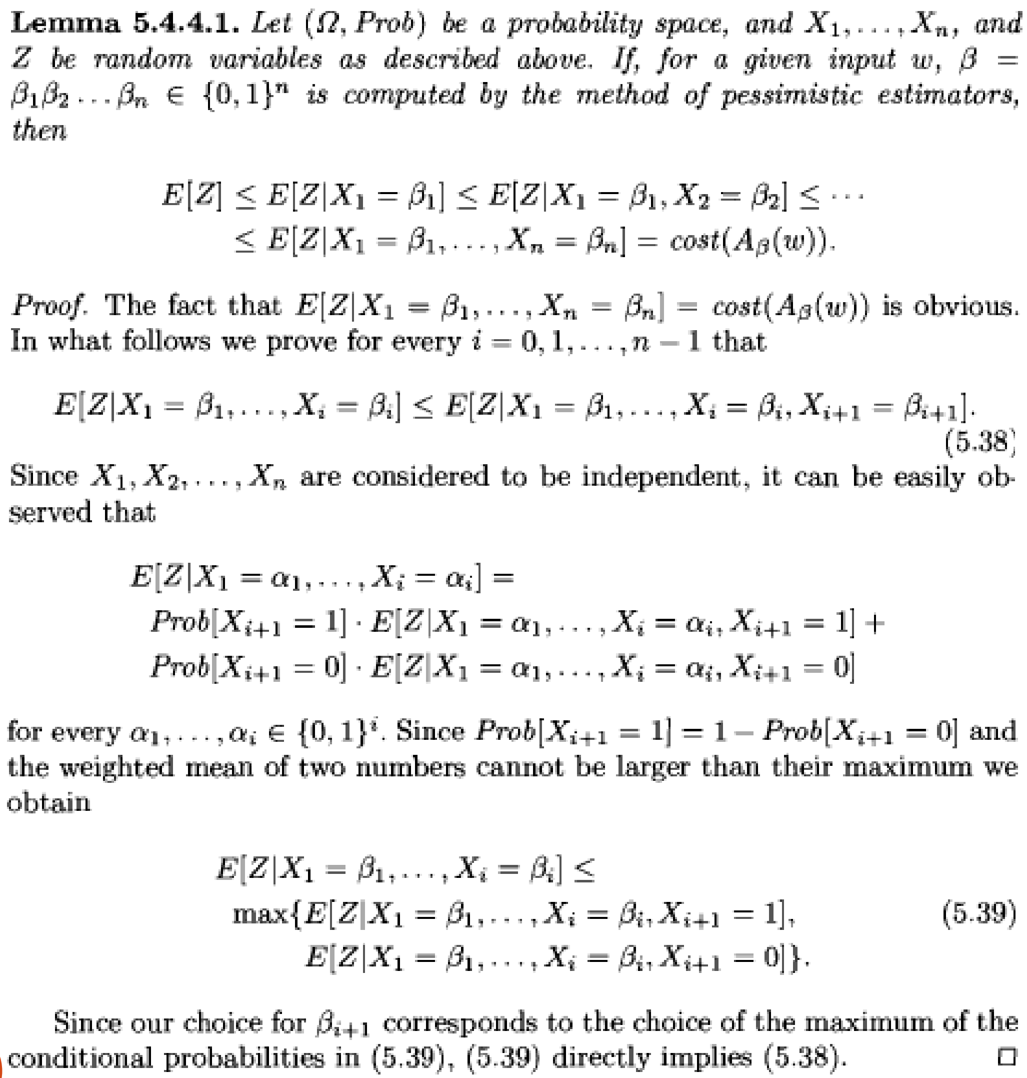

Finally, we observe that the derandomization method by the reduction of the size of the probability space is quite general and very powerful. But the complexity of the resulting deterministic algorithm may be too high. Already $O(n^4)$ for a formula in 3-CNF of n variables may be too large. Thus, from the practical point of view, the possibility of essentially reducing the number of random bits (choices) in a randomized algorithm may be the main current contribution of this method.

问题3： conditional probabilities

- 这种方法的基本思路是什么？
  - 主要面向什么样的问题？
  - 目标是什么？
  - 具体方法/算法是什么？为什么可以达成上述目标？
  - 算法中的关键步骤是什么？

- 对于MAX-E*k*SAT和RSMS是如何实现这一关键步骤的？

问题3： conditional probabilities (续)



Lemma 5.4.4.1. *Let $(\Omega, Prob)$ be a probability space, and $X_1, \dots, X_n$, and Z be random variables as described above. If, for a given input w, $\beta = \beta_1\beta_2\dots\beta_n \in \{0,1\}^n$ is computed by the method of pessimistic estimators, then*

$$\begin{aligned} E[Z] &\le E[Z|X_1 = \beta_1] \le E[Z|X_1 = \beta_1, X_2 = \beta_2] \le \cdots \\ &\le E[Z|X_1 = \beta_1, \dots, X_n = \beta_n] = cost(A_\beta(w)). \end{aligned}$$

Proof. The fact that $E[Z|X_1 = \beta_1, \dots, X_n = \beta_n] = cost(A_\beta(w))$ is obvious. In what follows we prove for every $i = 0, 1, \dots, n-1$ that

$$E[Z|X_1 = \beta_1, \dots, X_i = \beta_i] \le E[Z|X_1 = \beta_1, \dots, X_i = \beta_i, X_{i+1} = \beta_{i+1}]. \tag{5.38}$$

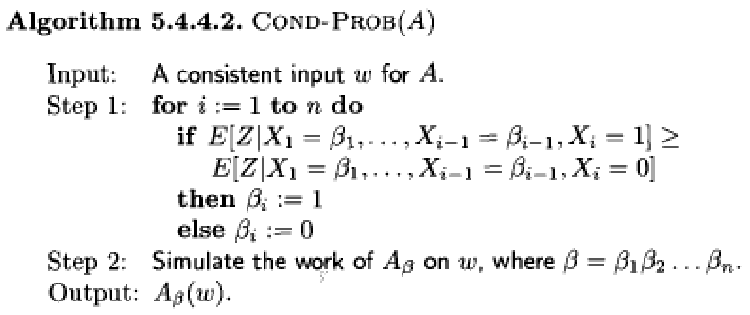
Since $X_1, X_2, \dots, X_n$ are considered to be independent, it can be easily observed that

$$\begin{aligned} &E[Z|X_1 = \alpha_1, \dots, X_i = \alpha_i] = \\ &\quad Prob[X_{i+1} = 1] \cdot E[Z|X_1 = \alpha_1, \dots, X_i = \alpha_i, X_{i+1} = 1] + \\ &\quad Prob[X_{i+1} = 0] \cdot E[Z|X_1 = \alpha_1, \dots, X_i = \alpha_i, X_{i+1} = 0] \end{aligned}$$

for every $\alpha_1, \dots, \alpha_i \in \{0,1\}^i$. Since $Prob[X_{i+1} = 1] = 1 - Prob[X_{i+1} = 0]$ and the weighted mean of two numbers cannot be larger than their maximum we obtain

$$\begin{aligned} &E[Z|X_1 = \beta_1, \dots, X_i = \beta_i] \le \\ &\quad \max\{E[Z|X_1 = \beta_1, \dots, X_i = \beta_i, X_{i+1} = 1], \\ &\qquad E[Z|X_1 = \beta_1, \dots, X_i = \beta_i, X_{i+1} = 0]\}. \end{aligned} \tag{5.39}$$

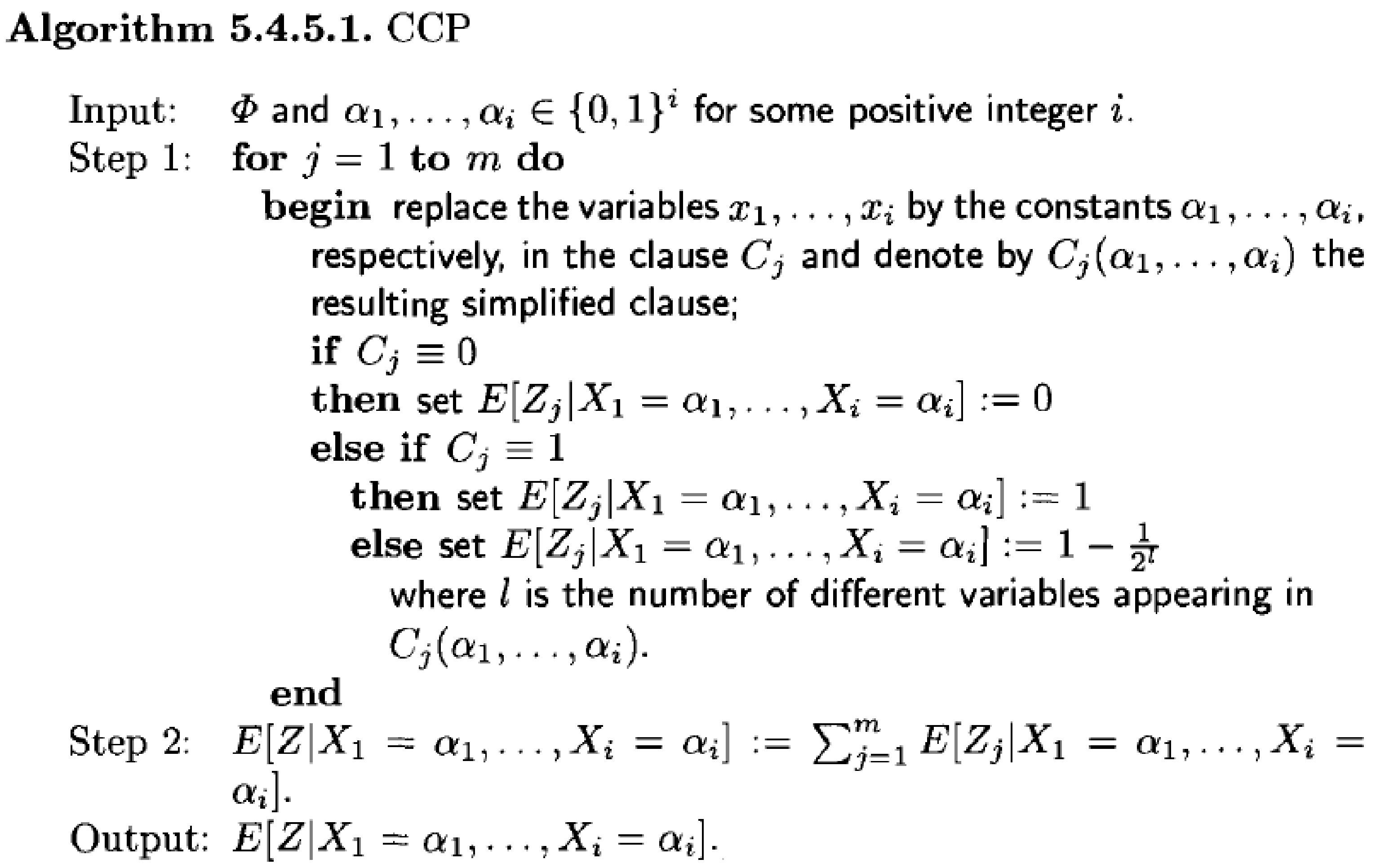
Since our choice for β_{i+1} corresponds to the choice of the maximum of the conditional probabilities in (5.39), (5.39) directly implies (5.38). □



Algorithm 5.4.4.2. COND-PROB(A)

```
Input:   A consistent input $w$ for $A$.
Step 1:  for $i := 1$ to $n$ do
           if $E[Z|X_1 = \beta_1, \dots, X_{i-1} = \beta_{i-1}, X_i = 1] \ge$
              $E[Z|X_1 = \beta_1, \dots, X_{i-1} = \beta_{i-1}, X_i = 0]$
           then $\beta_i := 1$
           else $\beta_i := 0$
Step 2:  Simulate the work of $A_\beta$ on $w$, where $\beta = \beta_1\beta_2\dots\beta_n$.
Output:  $A_\beta(w)$.
```

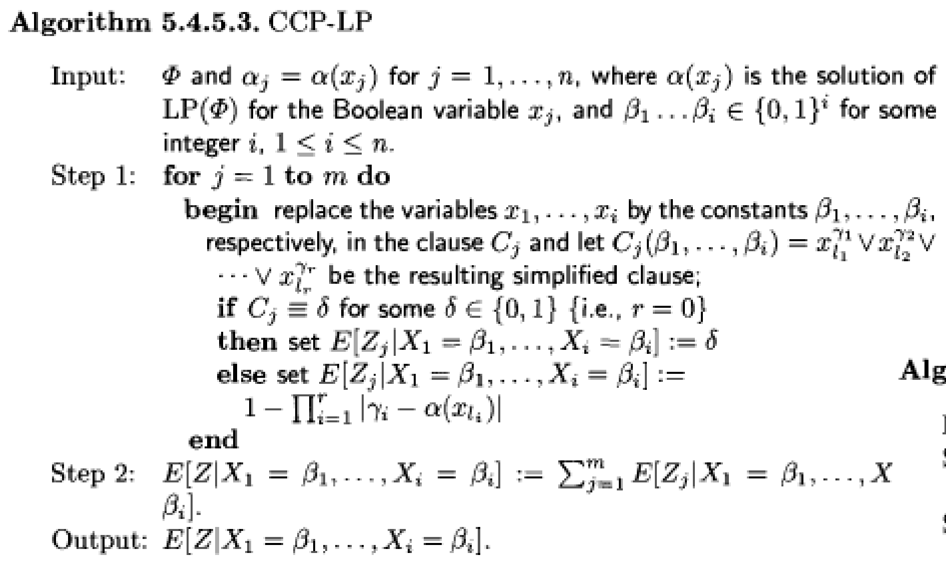
问题3： conditional probabilities (续)



Algorithm 5.4.5.1. CCP

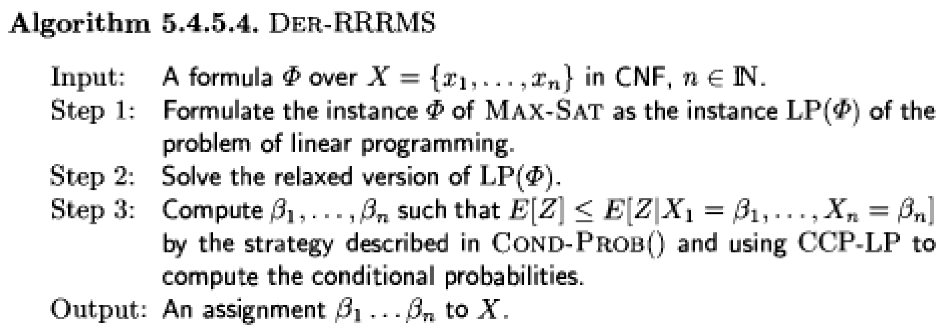
```
Input:   $\Phi$ and $\alpha_1, \ldots, \alpha_i \in \{0,1\}^i$ for some positive integer $i$.
Step 1:  for $j = 1$ to $m$ do
           begin replace the variables $x_1, \ldots, x_i$ by the constants $\alpha_1, \ldots, \alpha_i$,
             respectively, in the clause $C_j$ and denote by $C_j(\alpha_1, \ldots, \alpha_i)$ the
             resulting simplified clause;
             if $C_j \equiv 0$
             then set $E[Z_j | X_1 = \alpha_1, \ldots, X_i = \alpha_i] := 0$
             else if $C_j \equiv 1$
               then set $E[Z_j | X_1 = \alpha_1, \ldots, X_i = \alpha_i] := 1$
               else set $E[Z_j | X_1 = \alpha_1, \ldots, X_i = \alpha_i] := 1 - \frac{1}{2^l}$
                 where $l$ is the number of different variables appearing in
                 $C_j(\alpha_1, \ldots, \alpha_i)$.
           end
Step 2:  $E[Z | X_1 = \alpha_1, \ldots, X_i = \alpha_i] := \sum_{j=1}^{m} E[Z_j | X_1 = \alpha_1, \ldots, X_i =$
         $\alpha_i]$.
Output:  $E[Z | X_1 = \alpha_1, \ldots, X_i = \alpha_i]$.
```

问题3： conditional probabilities (续)



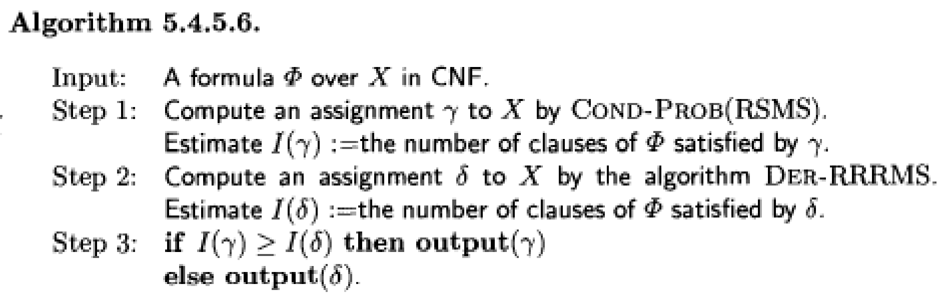
Algorithm 5.4.5.3. CCP-LP

```
Input:   $\Phi$ and $\alpha_j = \alpha(x_j)$ for $j = 1, \ldots, n$, where $\alpha(x_j)$ is the solution of
         LP($\Phi$) for the Boolean variable $x_j$, and $\beta_1 \ldots \beta_i \in \{0,1\}^i$ for some
         integer $i$, $1 \leq i \leq n$.
Step 1:  for $j = 1$ to $m$ do
           begin replace the variables $x_1, \ldots, x_i$ by the constants $\beta_1, \ldots, \beta_i$,
             respectively, in the clause $C_j$ and let $C_j(\beta_1, \ldots, \beta_i) = x_{l_1}^{\gamma_1} \vee x_{l_2}^{\gamma_2} \vee$
             $\cdots \vee x_{l_r}^{\gamma_r}$ be the resulting simplified clause;
             if $C_j \equiv \delta$ for some $\delta \in \{0,1\}$ {i.e., $r = 0$}
             then set $E[Z_j | X_1 = \beta_1, \ldots, X_i = \beta_i] := \delta$
             else set $E[Z_j | X_1 = \beta_1, \ldots, X_i = \beta_i] :=$
               $1 - \prod_{i=1}^{r} |\gamma_i - \alpha(x_{l_i})|$
           end
Step 2:  $E[Z | X_1 = \beta_1, \ldots, X_i = \beta_i] := \sum_{j=1}^{m} E[Z_j | X_1 = \beta_1, \ldots, X$
         $\beta_i]$.
Output:  $E[Z | X_1 = \beta_1, \ldots, X_i = \beta_i]$.
```



Algorithm 5.4.5.4. DER-RRRMS

```
Input:   A formula $\Phi$ over $X = \{x_1, \ldots, x_n\}$ in CNF, $n \in \mathbb{N}$.
Step 1:  Formulate the instance $\Phi$ of MAX-SAT as the instance LP($\Phi$) of the
         problem of linear programming.
Step 2:  Solve the relaxed version of LP($\Phi$).
Step 3:  Compute $\beta_1, \ldots, \beta_n$ such that $E[Z] \leq E[Z | X_1 = \beta_1, \ldots, X_n = \beta_n]$
         by the strategy described in COND-PROB() and using CCP-LP to
         compute the conditional probabilities.
Output:  An assignment $\beta_1 \ldots \beta_n$ to $X$.
```



Algorithm 5.4.5.6.

```
Input:   A formula $\Phi$ over $X$ in CNF.
Step 1:  Compute an assignment $\gamma$ to $X$ by COND-PROB(RSMS).
         Estimate $I(\gamma)$ :=the number of clauses of $\Phi$ satisfied by $\gamma$.
Step 2:  Compute an assignment $\delta$ to $X$ by the algorithm DER-RRRMS.
         Estimate $I(\delta)$ :=the number of clauses of $\Phi$ satisfied by $\delta$.
Step 3:  if $I(\gamma) \geq I(\delta)$ then output($\gamma$)
         else output($\delta$).
```