

# Linear Programming

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# Linear Programming

- 1 LP Forms
- 2 Primal and Dual
- 3 SSSP
- 4 Game

# Linear programming

Linear programming (LP):

max / min linear function  $f$  on  $x$

s.t.

linear constraints ( $\geq, =, \leq$ )

Mathematical programming:

- ▶ multi-objective
- ▶ non-linear objective/constraints
- ▶ integral variables

# Linear programming

$$\max \sum_{j=1}^n c_j x_j$$

s.t.

$$\sum_{j=1}^n a_{ij} x_j \leq b_i \quad i = 1 \dots m$$

$$x_j \geq 0 \quad j = 1 \dots n$$

$$\max \quad c^T x$$

s.t.

$$Ax \leq b$$

$$x \geq 0$$

$$\sum_{j=1}^n a_{ij} x_j \leq b_i \iff b_i - \sum_{j=1}^n a_{ij} x_j \geq 0$$

$$x_{n+i} = b_i - \sum_{j=1}^n a_{ij} x_j \quad x_{n+i} \geq 0$$

# Linear programming

(Problem: 29.1-4)

$$x_3 \leq 0$$

$$x_3 = x'_3 - x''_3 \quad x'_3, x''_3 \geq 0 \quad \times$$

$$x_3 = x'_3 - x''_3 \geq 0 \quad x'_3, x''_3 \geq 0 \quad \checkmark \text{ not necessary}$$

$$x_3 = -x_4 \quad x_4 \geq 0 \quad \checkmark$$

(Problem: 29.1-7)

$$\max x_1 - x_2$$

$$(x_1, x_2) = (\infty, 0)$$

Picture is NOT a proof!

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## Primal-dual

$$\max \quad c^T x$$

s.t.

$$Ax \leq b$$

$$x \geq 0$$

$$\min \quad b^T y$$

s.t.

$$A^T y \geq c$$

$$y \geq 0$$

## Primal-dual (Eq. 29.53)

$$\max \quad 3x_1 + x_2 + 2x_3$$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$

$$2x_1 + 2x_2 + 5x_3 \leq 24$$

$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, \quad x_2, \quad x_3 \geq 0$$

$$x^* = (8, 4, 0)$$

$$v^* = 28$$



# The multiplier approach

$$\textcircled{1} + \textcircled{2} \Rightarrow 3x_1 + 3x_2 + 8x_3 \leq 54$$

$$\textcircled{1} + \frac{1}{2} \times \textcircled{3} \Rightarrow 3x_1 + \frac{3}{2}x_2 + 4x_3 \leq 48$$

$$\textcircled{1} + \frac{1}{2} \times \textcircled{2} \Rightarrow 2x_1 + 2x_2 + \frac{11}{2}x_3 \quad \times$$

$$0 \times \textcircled{1} + \frac{1}{6} \times \textcircled{2} + \frac{2}{3} \times \textcircled{3} \Rightarrow 3x_1 + x_2 + \frac{13}{6}x_3 \leq 28$$

$$3x_1 + x_2 + 2x_3$$

$$\leq y_1 \times \textcircled{1} + y_2 \times \textcircled{2} + y_3 \times \textcircled{3}$$

$$= (y_1 + 2y_2 + 4y_3)x_1 + (y_1 + 2y_2 + y_3)x_2 + (3y_1 + 5y_2 + 2y_3)$$

$$\leq 30y_1 + 24y_2 + 36y_3$$

## Primal-dual (Eq. 29.86)

$$\max \quad 30y_1 + 24y_2 + 36y_3$$

s.t.

$$y_1 + 2y_2 + 4y_3 \geq 3$$

$$y_1 + 2y_2 + y_3 \geq 1$$

$$3y_1 + 5y_2 + 2y_3 \geq 2$$

$$y_1, \quad y_2, \quad y_3 \geq 0$$

$$y^* = \left(0, \frac{1}{6}, \frac{2}{3}\right) \quad v^* = 28$$

## Primal-dual (29.4-2)

$$\max \quad 3x_1 + x_2 + 2x_3$$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$

$$2x_1 + 2x_2 + 5x_3 \geq 24$$

$$4x_1 + x_2 + 2x_3 = 36$$

$$x_1 \geq 0$$

$$x_2 \leq 0$$

$$x_3 \begin{matrix} \geq \\ = \\ \leq \end{matrix} 0$$

$$\min \quad 30y_1 + 24y_2 + 36y_3$$

s.t.

$$y_1 + 2y_2 + 4y_3 \geq 3$$

$$y_1 + 2y_2 + y_3 \leq 1$$

$$3y_1 + 5y_2 + 2y_3 = 2$$

$$y_1 \geq 0$$

$$y_2 \leq 0$$

$$y_3 \begin{matrix} \geq \\ = \\ \leq \end{matrix} 0$$

# Weak/strong duality theorems

Theorem (Weak duality (29.8))

$$c^T x \leq b^T y \quad (\forall x, y)$$

Corollary (29.9)

$$c^T x \leq b^T y \Rightarrow x \text{ optimal to primal}; y \text{ optimal to dual.}$$

Theorem (Strong duality (29.10))

If an LP has a bounded optimal solution  $x^*$ , then

1. the dual has a bounded optimal solution  $y^*$
2.  $c^T x^* = b^T y^*$

**Remark:** P is unbounded  $\Rightarrow$  D is infeasible.

# Linear-inequality feasibility (29-1)

$$LP \Rightarrow LF$$

$$\max 0 \quad \text{😊}$$

$$\max -x_0 \quad (\text{Ch 29.5})$$

$$LF \Rightarrow LP$$

$$\max \quad c^T x$$

s.t.

$$Ax \leq b$$

$$x \geq 0$$

1. feasible?
2. unbounded?
3. finite optimal

# Linear-inequality feasibility

Binary search from  $c^T x = 0$ :

$$-2 \quad -1 \quad -\frac{1}{2} \quad \boxed{0} \quad \frac{1}{2} \quad 1 \quad 2 \quad 4$$

$$c^T x \geq 0 \begin{cases} c^T x \geq 2^0 \begin{cases} c^T x \geq 2^1 \\ c^T x \geq 2^{-1} \end{cases} \\ c^T x \geq -2^0 \begin{cases} c^T x \geq -2^{-1} \\ c^T x \geq -2^1 \end{cases} \end{cases}$$

Termination?  
Approximation?

# Linear-inequality feasibility

$$\max \quad c^T x$$

s.t.

$$Ax \leq b$$

$$x \geq 0$$

$$\min \quad b^T y$$

s.t.

$$A^T y \geq c$$

$$y \geq 0$$

$$b^T y \leq c^T x$$

$$Ax \leq b \quad A^T y \geq c$$

$$x \geq 0 \quad y \geq 0$$

**Remark:** What if this LF is infeasible?

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## SPSP (Ch 29.2)

$$\boxed{\max} \quad d_t$$

s.t.

$$d_v \leq d_u + w(u, v) \quad \forall (u, v) \in E$$

$$d_s = 0$$

$$Q_1 : \min d_t$$

$$Q_2 : d_v \geq 0 \quad \forall v \in V$$

$$Q_3 : d_v \leq d_u + w(u, v)$$

## SPSP

$$\min w(P)$$

s.t.

$$P : s \rightsquigarrow t$$

$$\min \sum_{(u,v) \in E} w_{uv} \cdot x_{uv}$$

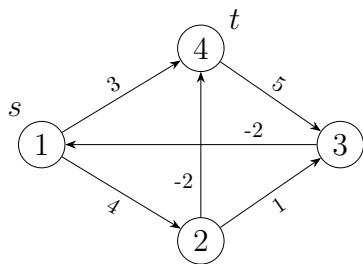
s.t.

$$P : s \rightsquigarrow t$$

$$x_{uv} = \{0, 1\} \quad \forall (u, v) \in E$$

$$\text{in}(v) - \text{out}(v) = \sum_u x_{uv} - \sum_u x_{vu} = \begin{cases} -1, & v = s \\ 1, & v = t \\ 0, & \text{o.w.} \end{cases}$$

## SPSP



$$\begin{pmatrix} -1 & -1 & & & 1 & & \\ 1 & & -1 & -1 & & & \\ & & 1 & 1 & -1 & & \\ & 1 & & 1 & & & -1 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{14} \\ x_{23} \\ x_{24} \\ x_{31} \\ x_{43} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

## SPSP

Dual: to max

$$\begin{aligned}
 \sum_{(u,v) \in E} w_{uv} \cdot x_{uv} &\stackrel{\geq}{=} (d_t - d_s)x_{12} + (d_t - d_s)x_{14} + \dots \\
 &= \sum_{(u,v) \in E} (d_v - d_u)x_{uv} \\
 &= d_t - d_s
 \end{aligned}$$

$$d_v - d_u \leq w(u, v) \iff d_v \leq d_u + w(u, v)$$

## SPSP: explanation

$$d_v \leq d_u + w(u, v) \quad \forall u : u \rightarrow v$$

$$\iff d_v \leq \min_{u:u \rightarrow v} d_u + w(u, v)$$

$$\xLeftrightarrow{\max d_v} d_v = \min_{u:u \rightarrow v} d_u + w(u, v)$$

Physical ball-string model: **PULL** it!

## SSSP (29.2-3)

$$\boxed{\max} \quad \sum_t d_t$$

s.t.

$$d_v \leq d_u + w(u, v) \quad \forall (u, v) \in E$$

$$d_s = 0$$

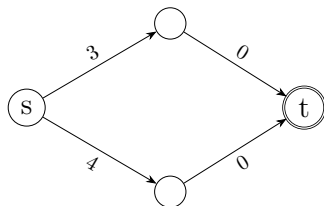
$$\max \sum_t d_t \iff \max\{d_t \mid t \in V\}$$

Proof.

- ▶ “ $\Rightarrow$ ”:
- ▶ “ $\Leftarrow$ ”:  $\max d_i$  never forces us to decrease  $d_j$ .



## SSSP (29.2-3)



Adding super node  $t$  and then running SPSP. ✗

## SSSP

$$\text{Dual: } \text{in}(v) - \text{out}(v) = \sum_u x_{uv} - \sum_u x_{vu} = \begin{cases} -1, & v = s \\ \boxed{1}, & v = t \\ 1, & \text{o.w.} \end{cases}$$

System of difference constraints  $\Rightarrow$  constraint graph  $\Rightarrow$  Bellman-Ford alg

## Highly Recommended

Simplex method vs. Dijkstra's alg & Bellman-Ford alg?

*"In an optimization problem, the identification of a dual problem is almost always coupled with the discovery of a polynomial time algorithm."*  
 — CLRS (Ch 29.4; 2nd)



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# Game

Alice: (e)conomy, (m)edicine

Bob: (t)ax, (i)mmigration

Votes: Alice gains vs. Bob loses

Strategy: pure vs. mixed

$$G = \begin{array}{c|cc} & t(y_1) & i(y_2) \\ \hline e(x_1) & 3 & -1 \\ m(x_2) & -2 & 1 \end{array}$$

$$\sum_{i,j} G_{ij} \cdot \mathbb{P}\{A_i, B_j\} = \sum_{i,j} G_{ij} x_i y_j$$

Q: Who Announce First?

## Game

$$\begin{array}{ll} \max & \boxed{\phantom{000}} \\ \text{s.t.} & \end{array}$$

$$x_1 + x_2 = 1$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$\begin{array}{ll} \min & \boxed{\phantom{000}} \\ \text{s.t.} & \end{array}$$

$$y_1 + y_2 = 1$$

$$y_1 \geq 0$$

$$y_2 \geq 0$$

$$z = \min\{3x_1 - 2x_2, -x_1 + x_2\}$$

$$\iff \begin{cases} \max z \\ z \leq 3x_1 - 2x_2 \\ z \leq -x_1 + x_2 \end{cases}$$

$$w = \max\{3y_1 - y_2, -2y_1 + y_2\}$$

$$\iff \begin{cases} \min w \\ w \geq 3y_1 - y_2 \\ w \geq -2y_1 + y_2 \end{cases}$$

## Game

$$\max \quad z$$

s.t.

$$-3x_1 + 2x_2 + z \leq 0$$

$$x_1 - x_2 + z \leq 0$$

$$x_1 + x_2 = 1$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$x^* = \left(\frac{3}{7}, \frac{4}{7}\right), \quad z = \frac{1}{7}$$

$$\min \quad w$$

s.t.

$$-3y_1 + y_2 + w \geq 0$$

$$2y_1 - y_2 + w \geq 0$$

$$y_1 + y_2 = 1$$

$$y_1 \geq 0$$

$$y_2 \geq 0$$

$$y^* = \left(\frac{2}{7}, \frac{5}{7}\right), \quad w = \frac{1}{7}$$

$$\max_x \min_y \sum_{i,j} G_{ij} x_i y_j \equiv \min_y \max_x \sum_{i,j} G_{ij} x_i y_j$$



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