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On the Solution of Linear Recurrence Equations

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Abstract. In this article, we present a general solution for linear divide-and-conquer recurrences of the form

$$u_n = \sum_{i=1}^{k} a_i u_{\lfloor \frac{n}{b_i} \rfloor} + g(n).$$

Our approach handles more cases than the Master method does[1]. We achieve this advantage by defining a new transform - the Order transform - which has useful properties for providing asymptotic answers (compared to other transforms which supply exact answers). This transform helps in mapping the sequence under consideration to the two dimensional plane where the solution becomes easier to obtain. We demonstrate the power of the final results by solving many "difficult" examples.

Keywords: Divide and conquer, Linear recurrence, Running time, Algorithm, Order transform, Order of growth.

1. Introduction

In this paper, we consider linear divide-and-conquer recurrences of the form:

$$u_n = \begin{cases} u_0 & n = 0 \\ \sum_{i=1}^{k} a_i u_{\lfloor \frac{n}{b_i} \rfloor} + g(n) & n \geq 1 \end{cases} \tag{1}$$

where:

- $u_0, a_i \in R^{*+}, \sum_{i=1}^{k} a_i \geq 1$
- $b_i, k \in N, b_i \geq 2, k \geq 1$
- $g(x)$ is defined for real values x, and is bounded, positive and nondecreasing function $\forall x \geq 0$
- $\forall c > 1, \exists x_1, k_1 > 0$ such that $g(\frac{x}{c}) \geq k_1 g(x), \forall x \geq x_1$

Such equations arise when studying the running time of divide-and-conquer algorithms. The Master method[1] addresses the problem for the case $k = 1$ only, with some restrictions on $g(n)$. The solution we present is valid $\forall k \geq 1$ and with minor restrictions on $g(n)$.

The main idea is to define a transform to help in mapping Equation 1 into a larger dimensional space where the order solution can be obtained easily. Consequently, we prove that if p_0 is the real solution of the characteristic equation

$$\sum_{i=1}^{k} a_i b_i^{-p} = 1$$

(which always exists and is unique and positive), then

$$u_n = \Theta(n^{p_0}) + \Theta\Big(n^{p_0} \int_{n_1}^{n} \frac{g(u)}{u^{p_0+1}} du\Big)$$

for n_1 large enough. In particular,

1. If $\exists\epsilon > 0$ such that $g(x) = O(x^{p_0-\epsilon})$, then $u_n = \Theta(n^{p_0})$.
2. If $\exists\epsilon > 0$ such that $g(x) = \Omega(x^{p_0+\epsilon})$ and $g(x)/x^{p_0+\epsilon}$ is a non decreasing function, then $u_n = \Theta(g(n))$.
3. If $g(x) = \Theta(x^{p_0})$, then $u_n = \Theta(n^{p_0} \log n)$.

Section 4 illustrates how to apply the above results to solve several interesting recurrence equations that cannot be handled by the Master method.

2. Literature Survey

According to Cormen, Leiserson, and Rivest [2], recurrences were studied as early as 1202 by L. Fibonacci, for whom the Fibonacci numbers are named. A. De Moivre introduced the method of generating functions for solving recurrences. The master method was provided by Bentley, Haken, and Saxe [1]. Knuth [3] and Liu [4] showed how to solve linear recurrences using the method of generating functions. Purdom and Brown [5] contains an extended discussion of recurrence solving. However, we are not aware of any work in the literature that solves the above divide-and-conquer linear recurrences, which are the subject of this paper.

3. The Solution

To determine the general form of the solution of Equation 1, we proceed as follows:

- In Theorem 1 we extend the domain of Equation 1 to the real line. The new "continuous" recurrence is described in Equation 2. We show that the two equations have isomorphic solutions.
- In Theorem 2 we define the Order transform and prove some of its interesting properties. We use this transform to extend the domain of Equation 2 to the two-dimensional plane.
- In Lemma 2 we show that the functions of interest do possess an order transform.
- In Theorem 3 we use the properties of the Order transform to solve Equation 2 in the plane.
- In Theorem 4 we show that our results agree with the Master method when the latter is applicable.

- In Corollary 1 we summarize the results as they apply for the original recurrence equation.

THEOREM 1 *Let u_n be a sequence defined as in Equation 1. Let $f(x)$ be a function defined by:*

$$f(x) = \begin{cases} u_0 & x \in [0,1) \\ \sum_{i=1}^{k} a_i f(\frac{x}{b_i}) + g(\lfloor x \rfloor) & x \in [1,\infty). \end{cases} \tag{2}$$

Then,

1. $\forall x \geq 0, f(x) = f(\lfloor x \rfloor)$.

2. $\forall n \geq 0, f(n) = u_n$.

In other words, $f(x)$ is a staircase function which matches with u_n at integer values of x. In proving the above theorem, we will use the following lemma whose proof can be found in [2].

LEMMA 1 *if $b \in N, b \geq 1$, and $x \in R^+$, then*

$$\left\lfloor \frac{x}{b} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{b} \right\rfloor.$$

Proof of Theorem 1: To prove that $f(x) = f(\lfloor x \rfloor)$, we use strong induction. Note that $\forall x \in [0,1)$ we have $f(x) = u_0$ and $f(\lfloor x \rfloor) = f(0) = u_0$. Hence, $f(\lfloor x \rfloor) = f(x)$. Now assume that $f(\lfloor x \rfloor) = f(x) \forall x \in [0,n)$, and let us prove that it is true $\forall x \in [n, n+1)$. Consider

$$f(x) = \sum_{i=1}^{k} a_i f\left(\frac{x}{b_i}\right) + g(\lfloor x \rfloor). \tag{3}$$

Let $x \in [n, n+1)$, then

$$\frac{x}{b_i} \in \left(0, \frac{n+1}{2}\right) \text{ since } b_i \geq 2.$$

But

$$\left(0, \frac{n+1}{2}\right) \subset [0,n) \text{ for } n \geq 1.$$

Hence, we conclude that

$$\frac{x}{b_i} \in [0,n), \frac{\lfloor x \rfloor}{b_i} \in [0,n) \text{ and } \left\lfloor \frac{\lfloor x \rfloor}{b_i} \right\rfloor = \left\lfloor \frac{x}{b_i} \right\rfloor \in [0,n) \text{ (using Lemma 1).}$$

Therefore,

$$\begin{aligned} f\Big(\frac{x}{b_i}\Big) &= f\Big(\Big\lfloor \frac{x}{b_i}\Big\rfloor\Big) \text{ (by assumption)} \\ &= f\Big(\Big\lfloor \frac{\lfloor x\rfloor}{b_i}\Big\rfloor\Big) \text{ (using Lemma 1)} \\ &= f\Big(\frac{\lfloor x\rfloor}{b_i}\Big) \text{ (by assumption).} \end{aligned}$$

Replacing in Equation 3, we obtain

$$f(x) = \sum_{i=1}^{k} a_i f\Big(\frac{\lfloor x\rfloor}{b_i}\Big) + g(\lfloor x\rfloor).$$

However,

$$f(\lfloor x\rfloor) = \sum_{i=1}^{k} a_i f\Big(\frac{\lfloor x\rfloor}{b_i}\Big) + g(\lfloor x\rfloor).$$

Therefore,

$$f(x) = f(\lfloor x\rfloor)\forall x \geq 0,$$

which completes the proof of Part 1 of the theorem.
To prove that $f(n) = u_n$, we use strong induction again.
The equality holds trivially for $n = 0$. Assume it is true for all $m < n$ and consider

$$f(n) = \sum_{i=1}^{k} a_i f\Big(\frac{n}{b_i}\Big) + g(n) \tag{4}$$

Let $n \geq 1$. We already proved in Part 1 that $f(n/b_i) = f(\lfloor n/b_i\rfloor)$. Now since $\lfloor n/b_i\rfloor \in [0, n)$ we conclude $f(\lfloor n/b_i\rfloor) = u_{\lfloor n/b_i\rfloor}$. Replacing in Equation 4, we obtain

$$f(n) = \sum_{i=1}^{k} a_i u_{\lfloor \frac{n}{b_i}\rfloor} + g(n).$$

But,

$$u_n = \sum_{i=1}^{k} a_i u_{\lfloor \frac{n}{b_i}\rfloor} + g(n).$$

So, $f(n) = u_n$, which completes the proof. ■

Definition 1. (Regularity Conditions) Let S be the set of all real-valued function $f(x)$ of the real variable x satisfying the following conditions:

1. $\forall x \geq 0, f(x)$ is bounded.

2. $\forall x \geq 0, f(x)$ is nondecreasing.

3. $\forall c > 1, \exists x_1, k_1 > 0$ such that $\forall x \geq x_1, f(\frac{x}{c}) \geq k_1 f(x)$,

In Lemma 2 we will show that the functions $f(x)$ and $g(\lfloor x \rfloor)$ as defined in Theorem 1 both belong to S.

THEOREM 2 *(The Order Transform) Let $P\{\}$ be a mapping that assigns to each function $f(x) \in S$ a real-valued function $F(s,p)$ of the real variables $s \in R^+$ and p, defined by:*

$$F(s,p) = P\{f(x)\} \equiv \int_1^s f(u)u^{-p-1}du.$$

Then $P\{\}$ satisfies the following properties:

1. *$P\{\}$ exists.*

2. *$P\{\}$ is linear.*

3. *$P\{\}$ is one-to-one.*

4. *(Scaling property) Let $f(x) \in S, F(s,p) = P\{f(x)\}, a \in R$ and $a > 1$. Then,*

$$P\Big\{f\Big(\frac{x}{a}\Big)\Big\} = a^{-p}F(s,p) - \Theta_s\Big(\frac{f(s)}{s^p}\Big) + \Theta_s(1),$$

where $\Theta_s(h(s,p))$ is a function bounded between $c_1(p)h(s,p)$ and $c_2(p)h(s,p)$, -for some positive functions $c_1(p), c_2(p)$-, $\forall s > s_0, \forall p$.

Proof:

1. Since f is bounded and the range of the integral is finite, then $P\{\}$ exists.

2. Linearity of the transform is trivial.

3. Let $f_1(x), f_2(x) \in S$ and let $P\{f_1(x)\} = P\{f_2(x)\}$. Then,

$$\begin{aligned} \int_1^s f_1(u)u^{-p-1}du &= \int_1^s f_2(u)u^{-p-1}du \\ \frac{\partial}{\partial s}\int_1^s f_1(u)u^{-p-1}du &= \frac{\partial}{\partial s}\int_1^s f_2(u)u^{-p-1}du \\ f_1(s)s^{-p-1} &= f_2(s)s^{-p-1} \\ f_1(s) &= f_2(s) \end{aligned}$$

which completes the proof that $P\{\}$ is one-to-one.

4. To prove the scaling property, let

$$F_1(s,p) = \int_1^s f\Big(\frac{u}{a}\Big)u^{-p-1}du$$

Making a change of variable $v = u/a$, we obtain

$$\begin{aligned} F_1(s,p) &= \int_{1/a}^{s/a} f(v)(av)^{-p-1}adv \\ &= a^{-p}\int_{1/a}^{s/a} f(v)v^{-p-1}dv \\ &= a^{-p}\Big[\int_1^s - \int_{s/a}^s + \int_{1/a}^1 f(v)v^{-p-1}dv\Big] \\ &= a^{-p}F(s,p) - a^{-p}\int_{s/a}^s f(v)v^{-p-1}dv \\ &\quad + a^{-p}\int_{1/a}^1 f(v)v^{-p-1}dv \end{aligned} \tag{5}$$

Note that the last term

$$a^{-p}\int_{1/a}^1 f(v)v^{-p-1}dv = \Theta_s(1). \tag{6}$$

Let us investigate the asymptotic behavior of $a^{-p}\int_{s/a}^s f(v)v^{-p-1}dv$ with respect to s. Since $f \in S$, then

(A) $\forall v > 0, f(v)$ is a non decreasing function, and

(B) $\exists k_1, s_1 > 0$ such that $k_1(v) \leq f(v/a)\ \forall v \geq s_1$.

Therefore, $\forall v \in [s/a, s]$ we have

$$f(s/a) \leq f(v) \leq f(s).$$

Since for $s \geq s_1$ we have $k_1 f(s) \leq f(s/a)$, then

$$\begin{aligned} k_1 f(s) &\leq f(v) \leq f(s)\ \forall s > s_1, \\ k_1\frac{f(s)}{v^{p+1}} &\leq \frac{f(v)}{v^{p+1}} \leq \frac{f(s)}{v^{p+1}}\ \forall s > s_1, \\ k_1 f(s)\int_{s/a}^s \frac{dv}{v^{p+1}} &\leq \int_{s/a}^s \frac{f(v)}{v^{p+1}}dv \leq f(s)\int_{s/a}^s \frac{dv}{v^{p+1}}\ \forall s > s_1. \end{aligned} \tag{7}$$

We have two cases to consider, the case of $p \neq 0$ and the case of $p = 0$.

(A) If $p \neq 0$, we get

$$\int_{s/a}^{s} \frac{dv}{v^{p+1}} = -\frac{1}{p}\Big[\frac{1}{v^p}\Big]_{s/a}^{s} = \frac{1}{s^p}\Big(\frac{a^p-1}{a^p}\Big).$$

Replacing in Equation 7, we obtain

$$k_1\frac{f(s)}{s^p}\Big(\frac{a^p-1}{p}\Big) \leq \int_{s/a}^{s}\frac{f(v)}{v^{p+1}}dv \leq \frac{f(s)}{s^p}\Big(\frac{a^p-1}{p}\Big)\ \forall s > s_1,$$

$$\text{i.e., } \int_{s/a}^{s}\frac{f(v)}{v^{p+1}}dv = \Theta_s\Big(\frac{f(s)}{s^p}\Big) \qquad \Big(\frac{a^p-1}{p} > 0,\ \text{since } a > 1\Big)$$

(B) If $p = 0$ then

$$\int_{s/a}^{s}\frac{dv}{v^{p+1}}dv = \log(s) - \log\left(\frac{s}{a}\right) = \log a = \frac{\log a}{s^p}.$$

Replacing in Equation 7, we obtain

$$k_1 \log a\frac{f(s)}{s^p} \leq \int_{s/a}^{s}\frac{f(v)}{v^{p+1}}dv \leq \log a\frac{f(s)}{s^p}\ \forall s > s_1,$$

$$\text{i.e., } \int_{s/a}^{s}\frac{f(v)}{v^{p+1}}dv = \Theta_s\Big(\frac{f(s)}{s^p}\Big) \qquad \text{(note that } \log a > 0). \tag{8}$$

Replacing Equations 6 and 8 in 5, we obtain

$$F_1(s,p) = a^{-p}F(s,p) - \Theta_s\Big(\frac{f(s)}{s^p}\Big) + \Theta_s(1),$$

which completes the proof.

■

LEMMA 2 *Let $f(x)$ be a function defined as in Theorem 1. In other words,*

$$f(x) = \begin{cases} u_0 & x \in [0,1) \\ \sum_{i=1}^{k} a_i f\Big(\lfloor\frac{x}{b_i}\rfloor\Big) + g(\lfloor x\rfloor) & x \in [1,\infty) \end{cases} \tag{9}$$

where:

- $u_0, a_i \in R^{*+}, \sum_{i=1}^{k} a_i \geq 1$
- $b_i, k \in N, b_i \geq 2, k \geq 1$

- $g(x)$ *is a bounded, positive and nondecreasing function* $\forall x \geq 0$
- $\forall c > 1, \exists x_1, k_1 > 0$ *such that* $g(\frac{x}{c}) \geq k_1 g(x), \forall x \geq x_1$.

Then,

1. $g(\lfloor x \rfloor) \in S$,
2. $f(x) \in S$ *(see footnote)*[1].

Proof:

1. Note that $g(x) \in S$ by definition. Hence, $g(\lfloor x \rfloor)$ is clearly bounded and non-decreasing. Moreover, $g(\lfloor x \rfloor)$ is positive. There remains to prove that

$$\forall c > 1, \exists x_2, k_2, \text{ such that } g(\lfloor \frac{x}{c} \rfloor) \geq k_2 g(\lfloor x \rfloor) \ \forall x \geq x_2.$$

Let $c > 1$ be given. Let $x > \max\{x_1 + c + 1, \frac{c^2}{c-1} + 1\}$, then the following three useful inequalities can be derived:

$$\begin{aligned} x &> x_1 + c + 1 \\ \lfloor x \rfloor &> x_1 + c \\ \lfloor x \rfloor - c &> x_1 \end{aligned} \tag{10}$$

On the other hand,

$$\begin{aligned} x &> \frac{c^2}{c-1} + 1 \\ \lfloor x \rfloor &> \frac{c^2}{c-1} \\ \lfloor x \rfloor (c-1) &> c^2 \\ \lfloor x \rfloor - \frac{\lfloor x \rfloor}{c} &> c \\ \lfloor x \rfloor - c &> \frac{\lfloor x \rfloor}{c} \end{aligned} \tag{11}$$

Finally,

$$\begin{aligned} x &> x_1 + c + 1 \\ \lfloor x \rfloor &> x_1 \end{aligned} \tag{12}$$

Using Inequalities 10, 11, and 12, we conclude

$$\begin{aligned} g\left(\left\lfloor \frac{x}{c} \right\rfloor\right) &= g\left(\left\lfloor \frac{\lfloor x \rfloor}{c} \right\rfloor\right) \text{ (using Lemma 1)} \\ &\geq g\left(\frac{\lfloor x \rfloor}{c} - 1\right) \text{ (since } g \text{ is non-decreasing)} \end{aligned}$$

$$\begin{aligned} &= g\Big(\frac{\lfloor x \rfloor - c}{c}\Big) \\ &\geq k_1 g(\lfloor x \rfloor - c) \text{ (using Inequality 10)} \\ &> k_1 g\Big(\frac{\lfloor x \rfloor}{c}\Big) \text{ (using Inequality 11)} \\ &> {k_1}^2 g(\lfloor x \rfloor) \text{ (using Inequality 12).} \end{aligned}$$

Therefore, $\forall c > 1, \exists x_2 = \max(x_1 + c + 1, \frac{c^2}{c-1} + 1) > 0, k_2 = {k_1}^2 > 0$, such that $g(\lfloor \frac{x}{c} \rfloor) \geq k_2 g(\lfloor x \rfloor)$, which completes the proof that $g(\lfloor x \rfloor) \in S$.

2. To prove that $f(x) \in S$, note that f is defined in terms of a finite sum of positive terms each of which is bounded and also positive. So, f is bounded and positive. Also according to Theorem 1, $f(x) = f(\lfloor x \rfloor)$. So, it is sufficient to prove that

   (A) $f(n)$ is non-decreasing for all $n > 0$,

   (B) $\forall c > 1, \exists n_3, k_3$ such that $f(\frac{n}{c}) \geq k_3 f(n)\ \forall n > n_3$.

   To prove that $f(n)$ is non-decreasing, we use strong induction. Note that

$$\begin{aligned} f(0) &= u_0, \\ f(1) &= \sum_{i=1}^{k} a_i u_0 + g(1) \geq u_0 \qquad \text{(since } \textstyle\sum_{i=1}^{k} a_i \geq 1 \text{ and } g(1) \geq 0) \end{aligned}$$

   Assume that for all $k < n$ we have $f(k) \geq f(k-1)$, and let us prove that $f(n) \geq f(n-1)$. Consider

$$\begin{aligned} f(n) &= \sum_{i=1}^{k} a_i f\Big(\frac{n}{b_i}\Big) + g(n) \\ &= \sum_{i=1}^{k} a_i f\Big(\Big\lfloor \frac{n}{b_i} \Big\rfloor\Big) + g(n) \\ &\geq \sum_{i=1}^{k} a_i f\Big(\Big\lfloor \frac{n-1}{b_i} \Big\rfloor\Big) + g(n-1) \qquad (g(n) \text{ is nondecreasing}) \\ &= \sum_{i=1}^{k} a_i f\Big(\frac{n-1}{b_i}\Big) + g(n-1) \\ &= f(n-1). \end{aligned}$$

   So, for all $n \geq 1$ we have $f(n) \geq f(n-1)$. Therefore, f is non-decreasing.
   To prove the other regularity condition, we use strong induction again. Using the results of the previous Part, $\exists k_2, x_2$, such that $g(\lfloor \frac{x}{c} \rfloor) \geq k_2 g(\lfloor x \rfloor)\ \forall x \geq x_2, \forall c > 1$. Consider

$$\begin{aligned} n_0 &= \lfloor x_2 \rfloor, \\ k_3 &= \min\Big\{\frac{f(0)}{f(n_0)}, k_2\Big\}. \end{aligned}$$

f is nondecreasing and strictly positive. So, $\forall m \in [0, n_0]$

$$\begin{aligned} f\Big(\frac{m}{c}\Big) &\geq f(0) \\ &\geq f(0)\frac{f(m)}{f(n_0)} \\ &= f(m)\frac{f(0)}{f(n_0)} \\ &\geq f(m)k_3 \end{aligned}$$

i.e., $\forall m \in [0, n_0]$ we have $f(\frac{m}{c}) \geq k_3 f(m)$. Now assume that for all $m \in [0, n)$ where $n > n_0$ we have $f(\frac{m}{c}) \geq k_3 f(m)$, and let us prove that $f(\frac{n}{c}) \geq k_3 f(n)$. Since $n > n_0$, we have $n/c > 1$. So,

$$\begin{aligned} f\Big(\frac{n}{c}\Big) &= \sum_{i=1}^{k} f\Big(\frac{n/a_i}{c}\Big) + g\Big(\Big\lfloor\frac{n}{c}\Big\rfloor\Big) \\ &= \sum_{i=1}^{k} f\Big(\Big\lfloor\frac{n/a_i}{c}\Big\rfloor\Big) + g\Big(\Big\lfloor\frac{n}{c}\Big\rfloor\Big) \qquad \text{(since } f(n) = f(\lfloor n\rfloor)) \text{ ,} \\ &= \sum_{i=1}^{k} f\Big(\Big\lfloor\frac{\lfloor n/a_i\rfloor}{c}\Big\rfloor\Big) + g\Big(\Big\lfloor\frac{n}{c}\Big\rfloor\Big) \qquad \text{(since } \Big\lfloor\frac{n}{c}\Big\rfloor = \Big\lfloor\frac{\lfloor n\rfloor}{c}\Big\rfloor\text{),} \\ &= \sum_{i=1}^{k} f\Big(\frac{\lfloor n/a_i\rfloor}{c}\Big) + g\Big(\Big\lfloor\frac{n}{c}\Big\rfloor\Big) \qquad \text{(since } f(n) = f(\lfloor n\rfloor)\text{),} \\ &\geq \sum_{i=1}^{k} k_3 f\Big(\Big\lfloor\frac{n}{a_i}\Big\rfloor\Big) + k_3 g(n) \qquad \text{(since } \Big\lfloor\frac{n}{a_i}\Big\rfloor < n \text{ and } g(n) \in S\text{),} \\ &= \sum_{i=1}^{k} k_3 f\Big(\frac{n}{a_i}\Big) + k_3 g(n) \qquad \text{(since } f(n) = f(\lfloor n\rfloor)\text{),} \\ &= k_3 f(n). \end{aligned}$$

As a result,

$$\forall c > 1, \exists n_3 = 0, k_3 = \min\Big\{\frac{f(0)}{f(n_0)}, k_2\Big\}, \text{ such that } f\Big(\frac{n}{c}\Big) \geq k_3 f(n)\ \forall n \geq n_3,$$

which completes the proof. ■

THEOREM 3 *Let $f(x)$ be a function defined as in Theorem 1. Let p_0 be the real solution of the characteristic equation $\sum_{i=1}^{k} a_i b_i^{-p} = 1$. Then p_0 always exists and is unique and positive. Furthermore,*

$$f(x) = \Theta(x^{p_0}) + \Theta\Big(x^{p_0}\int_{x_1}^{x}\frac{g(u)}{u^{p_0+1}}du\Big)$$

for x_1 large enough.

Proof:

Since $g(x) \in S$, then according to Lemma 1 both $g(\lfloor x \rfloor)$ and $f(x)$ belong to S. Hence, both functions possess an Order transform. Rewriting the definition of $f(x)$,

$$\begin{aligned}
f(x) &= \sum_{i=1}^{k} a_i f\Big(\frac{x}{b_i}\Big) + g(\lfloor x \rfloor) \qquad \forall x > 1 \\
P\{f(x)\} &= P\Big\{\sum_{i=1}^{k} a_i f\Big(\frac{x}{b_i}\Big) + g(\lfloor x \rfloor)\Big\} \qquad \forall s > 1 \\
P\{f(x)\} &= \sum_{i=1}^{k} a_i P\Big\{f\Big(\frac{x}{b_i}\Big)\Big\} + \int_1^s \frac{g(\lfloor u \rfloor)}{u^{p+1}} du \\
F(s,p) &= \sum_{i=1}^{k} a_i \Big(b_i^{-p} F(s,p) - \Theta_s\Big(\frac{f(s)}{s^p}\Big) + \Theta_s(1)\Big) + \int_1^s \frac{g(\lfloor u \rfloor)}{u^{p+1}} du
\end{aligned}$$

$$\text{i.e., } F(s,p)\Big(1 - \sum_{i=1}^{k} a_i b_i^{-p}\Big) + \Theta_s\Big(\frac{f(s)}{s^p}\Big) = \int_1^s \frac{g(\lfloor u \rfloor)}{u^{p+1}} du + \Theta_s(1). \tag{13}$$

Let $h(p) = 1 - \sum_{i=1}^{k} a_i b_i^{-p}$. Then,

$$\begin{aligned}
h(0) &= 1 - \sum_{i=1}^{k} a_i \leq 0, \\
\lim_{p \to \infty} h(p) &= 1 > 0, \\
\frac{d}{dp} h(p) &= \sum_{i=1}^{k} a_i (\log b_i) b_i^{-p} > 0, \forall p \qquad (b_i \geq 2, a_i > 0).
\end{aligned}$$

So, $h(p) = 0$ has a unique positive solution p_0. Replacing p_0 in Equation 13, we get

$$\Theta_s\Big(\frac{f(s)}{s^{p_0}}\Big) = \int_1^s \frac{g(\lfloor u \rfloor)}{u^{p_0+1}} du + \Theta_s(1)$$

$$\text{i.e., } f(x) = \Theta\Big(x^{p_0} \int_1^x \frac{g(\lfloor u \rfloor)}{u^{p_0+1}}\Big) du + \Theta(x^{p_0}). \tag{14}$$

Since $g(x) \in S$ and $g(x)$ is non-decreasing, then

$$\forall x \geq 1, g\Big(\frac{x}{2}\Big) \leq g(\lfloor x \rfloor) \leq g(x), \text{ and}$$

$$\exists k_1, x_1 > 0 \text{ such that } g\Big(\frac{x}{2}\Big) \geq k_1 g(x) \forall x > x_1.$$

In other words, $\forall x > x1, k_1 g(x) < g(\lfloor x \rfloor) < g(x)$,

$$\text{i.e., } k_1 \int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} du \leq \int_{x_1}^{x} \frac{g(\lfloor u \rfloor)}{u^{p_0+1}} du \leq \int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} du$$

$$\text{i.e., } x^{p_0} \int_{x_1}^{x} \frac{g(\lfloor u \rfloor)}{u^{p_0+1}} du = \Theta\Big(x^{p_0} \int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} du\Big).$$

Replacing in Equation 14 we obtain

$$f(x) = \Theta\Big(x^{p_0} \int_{1}^{x_1} \frac{g(\lfloor u \rfloor)}{u^{p_0+1}} du\Big) + \Theta\Big(x^{p_0} \int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} du\Big) + \Theta(x^{p_0})$$

$$\text{i.e., } f(x) = \Theta(x^{p_0}) + \Theta\Big(x^{p_0} \int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} du\Big),$$

which completes the proof. ■

THEOREM 4 *Let $f(x)$ be a function defined as in Theorem 1. Let p_0 be the unique solution of the characteristic equation. Then,*

1. *If $\exists \epsilon > 0$ such that $g(x) = O(x^{p_0-\epsilon})$, then $f(x) = \Theta(x^{p_0})$.*
2. *If $\exists \epsilon > 0$ such that $g(x) = \Omega(x^{p_0+\epsilon})$ and $g(x)/x^{p_0+\epsilon}$ is a non decreasing function, then $f(x) = \Theta(g(x))$.*
3. *If $g(x) = \Theta(x^{p_0})$ then $f(x) = \Theta(x^{p_0} \log x)$.*

Proof:

1. Suppose $\exists \epsilon > 0$ such that $g(x) = O(x^{p_0-\epsilon})$, i.e.,

$$\exists x_0, k > 0, \text{ such that } g(x) < kx^{p_0-\epsilon}, \forall x > x_0.$$

Let $x_1 > x_0$, then $\forall x > x_1$ we have

$$\begin{aligned} \int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} du &\leq k \int_{x_1}^{x} \frac{ku^{p_0-\epsilon}}{u^{p_0+1}} du \\ &= k \int_{x_1}^{x} \frac{du}{u^{p_0+1}} \\ &= \frac{k}{\epsilon}\Big(\frac{1}{x_1^{\epsilon}} - \frac{1}{x^{\epsilon}}\Big) \\ &< \frac{k}{\epsilon}\frac{1}{x_1^{\epsilon}} \\ &= O(1) \end{aligned} \tag{15}$$

But we proved in Theorem 3 that

$$f(x) = \Theta(x^{p_0}) + \Theta\Big(x^{p_0}\int_{x_1}^{x}\frac{g(u)}{u^{p_0+1}}du\Big) \tag{16}$$

Replacing Equation 16 in 15 we obtain

$$f(x) = \Theta(x^{p_0}) + O(x^{p_0}) = \Theta(x^{p_0})$$

which completes the proof.

2. Suppose $\exists\epsilon > 0$ such that $g(x) = \Omega(x^{p_0+\epsilon})$ and $g(x)/x^{p_0+\epsilon}$ is a non-decreasing function for large x. Let $\phi(x) = g(x)/x^{p_0+\epsilon}$. Then,

$$g(x) = \phi(x)x^{p_0+\epsilon}$$

$$\text{i.e., } x^{p_0}\int_{x_1}^{x}\frac{g(u)}{u^{p_0+1}}du = x^{p_0}\int_{x_1}^{x}\phi(u)u^{\epsilon-1}du$$

for x_1 large enough to make $\phi(x)$ non-decreasing. Therefore, $\forall x > x_1$

$$\begin{aligned} x^{p_0}\int_{x_1}^{x}\frac{g(u)}{u^{p_0+1}}du &< x^{p_0}\phi(x)\int_{x_1}^{x}u^{\epsilon-1}du \\ &= x^{p_0}\phi(x)\frac{1}{\epsilon}(x^{\epsilon} - x_1^{\epsilon}) \\ &< x^{p_0}\phi(x)\frac{1}{\epsilon}x^{\epsilon} \\ &= \frac{1}{\epsilon}g(x) \\ &= O(g(x)) \end{aligned} \tag{17}$$

Furthermore,

$$x^{p_0}\int_{x_1}^{x}\frac{g(u)}{u^{p_0+1}}du = x^{p_0}\int_{x/a}^{x}\frac{g(u)}{u^{p_0+1}}du + x^{p_0}\int_{x_1}^{x/a}\frac{g(u)}{u^{p_0+1}}du.$$

But $g(x) \in S$, so

$$\int_{x/a}^{x}\frac{g(u)}{u^{p_0+1}}du = \Theta\Big(\frac{g(x)}{x^{p_0}}\Big) \text{ (from proof of scaling property)}$$

Therefore,

$$\begin{aligned} x^{p_0}\int_{x_1}^{x}\frac{g(u)}{u^{p_0+1}}du &= \Theta(g(x)) + x^{p_0}\int_{x_1}^{x/a}\frac{g(u)}{u^{p_0+1}}du \\ &= \Omega(g(x)) \end{aligned} \tag{18}$$

Equations 17 and 18 imply that:

$$x^{p_0} \int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} du = \Theta(g(x)) \tag{19}$$

But we proved in Theorem 3 that

$$f(x) = \Theta(x^{p_0}) + \Theta\Big(x^{p_0} \int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} du\Big)$$

replacing Equation 19 in the above equation we obtain

$$\begin{aligned} f(x) &= \Theta(x^{p_0}) + \Theta(g(x)) \\ &= \Theta(g(x)) \qquad \text{since } g(x) = \Omega(x^{p_0+\epsilon}), \end{aligned}$$

which completes the proof.

3. Suppose $g(x) = \Theta(x^{p_0})$. Replacing in the result of Theorem 3, we get

$$\begin{aligned} f(x) &= \Theta(x^{p_0} \int_{x_1}^{x} \frac{du}{u}) + \Theta(x^{p_0}) \\ &= \Theta(x^{p_0} \log x), \end{aligned}$$

which completes the proof.

■

COROLLARY 1 *Let u_n be a sequence defined as in Equation 1. Then,*

$$u_n = \Theta(n^{p_0}) + \Theta\Big(n^{p_0} \int_{n_1}^{n} \frac{g(u)}{u^{p_0+1}} du\Big) \textit{ for } n_1 \textit{ large enough,}$$

where p_0 is the real solution of the equation $\sum_{i=1}^{k} a_i b_i^{-p} = 1$ which always exists and is unique and positive. Furthermore,

1. *if $\exists \epsilon > 0$ such that $g(x) = O(x^{p_0-\epsilon})$ then $u_n = \Theta(n^{p_0})$.*
2. *If $\exists \epsilon > 0$ such that $g(x) = \Omega(x^{p_0+\epsilon})$, and $g(x)/x^{p_0+\epsilon}$ is a non-decreasing function, then $u_n = \Theta(g(n))$.*
3. *If $g(x) = \Theta(x^{p_0})$ then $u_n = \Theta(n^{p_0} \log n)$.*

Proof: The proof can be quickly obtained by combining the results of Theorems 1, 2, 3, and 4. ■

4. Illustrative Examples

In this section we present five examples illustrating how to apply the above results.

EXAMPLE: $u_n = 2u_{\lfloor \frac{n}{2} \rfloor} + \Theta(n \log^2 \log n)$

Solving the characteristic equation, we get

$$\begin{aligned} 2 \times 2^{-p_0} &= 1 \\ p_0 &= 1. \end{aligned}$$

So,

$$\begin{aligned} u_n &= \Theta(n) + \Theta\Big(n \int_{n_1}^{n} \frac{u \log^2 \log u}{u^2} du\Big) \\ &= \Theta(n) + \Theta(n \log u(2 - 2 \log\log u - \log^2 \log u)]_{n_1}^{n}) \\ &= \Theta(n \log \log^2 \log n). \end{aligned}$$

□

EXAMPLE: $u_n = 2u_{\lfloor \frac{n}{3} \rfloor} + 1.5u_{\lfloor \frac{n}{4} \rfloor} + 5u_{\lfloor \frac{n}{2} \rfloor} + \Theta(n^2)$

Solving the characteristic equation, we get

$$\begin{aligned} 2 \times 3^{-p_0} + 1.5 \times 4^{-p_0} + 5 \times 2^{-p_0} &= 1 \\ p_0 &= 2.57450. \end{aligned}$$

Since $\exists \epsilon > 0$ such that $x^2 = O(x^{2.57450-\epsilon})$, then $u_n = \Theta(n^{2.5745})$. □

EXAMPLE: $u_n = 2u_{\lfloor \frac{n}{5} \rfloor} + u_{\lfloor \frac{n}{6} \rfloor} + \Theta(n^2)$

Solving the characteristic equation, we get

$$\begin{aligned} 2 \times 5^{-p_0} + 6^{-p_0} &= 1 \\ p_0 &= 0.678670. \end{aligned}$$

Since $\exists \epsilon > 0$ such that $x^2 = \Omega(x^{0.678670+\epsilon})$, and $x^2/n^{0.678670+\epsilon}$ is non-decreasing, then $u_n = \Theta(n^2)$. □

EXAMPLE: $u_n = \frac{4}{3}u_{\lfloor \frac{n}{2} \rfloor} + 3u_{\lfloor \frac{n}{3} \rfloor} + \frac{16}{3}u_{\lfloor \frac{n}{4} \rfloor} + \Theta(n^2 \log\log n)$

Solving the characteristic equation, we get

$$\begin{aligned} \frac{4}{3}2^{-p_0} + 3 \times 3^{-p_0} + \frac{16}{3}4^{-p_0} &= 1 \\ p_0 &= 2. \end{aligned}$$

So,

$$\begin{aligned} u_n &= \Theta(n^2) + \Theta\Big(n^2 \int_1^n \frac{u^2 \log\log u}{u^3} du\Big) \\ &= \Theta(n^2) + \Theta(n^2 \log n \log\log n) \\ &= \Theta(n^2 \log n \log\log n). \end{aligned}$$

□

EXAMPLE: $$u_n = \frac{3}{4} u_{\lfloor \frac{n}{2} \rfloor} + u_{\lfloor \frac{n}{3} \rfloor} + u_{\lfloor \frac{n}{6} \rfloor} + u_{\lfloor \frac{n}{8} \rfloor} + \Theta(n)$$

Solving the characteristic equation, we get

$$\begin{aligned} \frac{3}{4} 2^{-p} + 3^{-p} + 6^{-p} + 8^{-p} &= 1 \\ p_0 &= 1. \end{aligned}$$

Since $g(x) = \Theta(x) = \Theta(x^{p_0})$, then $u_n = \Theta(n \log n)$. □

5. Conclusion

In this article we provided a general method for solving linear divide-and-conquer recurrences. The solution turned out to have an integral form. The solution includes a parameter p_0 which is the root of the recurrence characteristic equation. The root can be computed by simple numerical algorithms.

Notes

1. The proof of this Lemma is quite lengthy and may be skipped at first reading.

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