2-2 The Efficiency of Algorithms

Hengfeng Wei

hfwei@nju.edu.cn

March 05, 2020
Donald E. Knuth (1938 ~)
Donald E. Knuth (1974)
Donald E. Knuth (1974)

“For his major contributions to the analysis of algorithms and the design of programming languages, and in particular for his contributions to the “art of computer programming” through his well-known books in a continuous series by this title.”
“People who analyze algorithms have double happiness."
“People who analyze algorithms have double happiness. First of all they experience the sheer beauty of elegant mathematical patterns that surround elegant computational procedures.”
Fibonacci numbers in the analysis of Euclid’s GCD algorithm

“People who analyze algorithms have double happiness. First of all they experience the sheer beauty of elegant mathematical patterns that surround elegant computational procedures."
Fibonacci numbers in the analysis of Euclid’s GCD algorithm $H_n$ in the analysis of FIND-MAX @ Stanford Lecture by Knuth

“People who analyze algorithms have double happiness. First of all they experience the sheer beauty of elegant mathematical patterns that surround elegant computational procedures.
“People who analyze algorithms have double happiness. First of all they experience the sheer beauty of elegant mathematical patterns that surround elegant computational procedures. Then they receive a practical payoff when their theories make it possible to get other jobs done more quickly and more economically.”
How Fast is It?
How Fast is It?

Time (and Space) Complexity of Algorithms
How Fast is It?

Time (and Space) Complexity of Algorithms

\[ \Omega \quad O \quad \Theta \quad o \quad \omega \]
Space Complexity of Algorithms
Space Complexity of Algorithms

We only care about the extra space caused by the algorithm.
Space Complexity of Algorithms

We only care about the extra space caused by the algorithm. The space for inputs is not part of space complexity of algorithms.
Space Complexity of Algorithms

We only care about the extra space caused by the algorithm. The space for inputs is not part of space complexity of algorithms.

\[
\text{INSERTION-SORT}(A, n) : O(1) \quad \text{(constant)}
\]
Is it the Fastest?
Is it the Fastest?

Complexity of Problems
Is it the Fastest?

Complexity of Problems

This is much harder and is not our focus today.
Whenever you design an algorithm, you provide an upper bound for the complexity of the problem. Whenever you encounter a "hardcore" of the problem, you obtain a lower bound for all possible algorithms. Often, there is an "algorithmic gap" between them. When the gap is gone, you get the optimal algorithm.

\[ \text{sorting} (A, n) : \Theta(n \log n) = O(n \log n) \cap \Omega(n \log n) \]

Hengfeng Wei (hfwei@nju.edu.cn)
Whenever you design an algorithm,
Whenever you design an algorithm, you provide an upper bound for the complexity of the problem.
Whenever you design an algorithm, you provide an upper bound for the complexity of the problem.

Whenever you encounter a “hardcore” of the problem,
Whenever you design an algorithm, you provide an upper bound for the complexity of the problem.

Whenever you encounter a “hardcore” of the problem, you obtain a lower bound for all possible algorithms.
Whenever you design an algorithm, you provide an upper bound for the complexity of the problem. Whenever you encounter a “hardcore” of the problem, you obtain a lower bound for all possible algorithms. Often, there is an “algorithmic gap” between them.
Whenever you design an algorithm, you provide an upper bound for the complexity of the problem.

Whenever you encounter a “hardcore” of the problem, you obtain a lower bound for all possible algorithms.

Often, there is an “algorithmic gap” between them. When the gap is gone, you get the optimal algorithm.
Whenever you design an algorithm, you provide an upper bound for the complexity of the problem.

Whenever you encounter a “hardcore” of the problem, you obtain a lower bound for all possible algorithms.

Often, there is an “algorithmic gap” between them.

When the gap is gone, you get the optimal algorithm.

\[
\text{sorting}(A, n) : \Theta(n \log n) = O(n \log n) \cap \Omega(n \log n)
\]
$Q$: How fast is your algorithm?
Q : How fast is your algorithm?

A : It runs 3.1415926 seconds.
Disadvantages:
Disadvantages:

- On different machines
Disadvantages:

- On different machines
- At different time
Disadvantages:

▶ On different machines
▶ At different time
▶ On different inputs
Disadvantages:

▶ On different machines
▶ At different time
▶ On different inputs
Disadvantages:

- On different machines
- At different time
- On different inputs

No Standards.
We need a uniform model of computation.
We need a uniform model of computation.

The RAM (Random Access Machine) Model of Computation
The RAM (Random Access Machine) Model of Computation

- Each memory access takes constant time.
- Each “primitive” operation takes constant time.
- Compound operations should be decomposed.
The RAM (Random Access Machine) Model of Computation

- Each memory access takes constant time.
- Each “primitive” operation takes constant time.
- Compound operations should be decomposed.

Counting up the number of time units.
Disadvantages:

- On different machines
- At different time
- On different inputs
Disadvantages:

- On different machines
- At different time
- On different inputs

Counting up the number of time units as a function of the input size in typical cases.
**Insertion-Sort (A)**

1. **for** $j = 2$ to $A.length$
2.  \hspace{1cm} key = $A[j]$
3.  \hspace{1cm} // Insert $A[j]$ into the sorted sequence $A[1..j-1]$.
4.  \hspace{1cm} $i = j - 1$
5.  \hspace{1cm} **while** $i > 0$ and $A[i] > key$
6.  \hspace{2cm} $A[i+1] = A[i]$
7.  \hspace{1cm} $i = i - 1$
8.  \hspace{1cm} $A[i+1] = key$

<table>
<thead>
<tr>
<th>cost</th>
<th>times</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$n$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$c_4$</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$c_5$</td>
<td>$\sum_{j=2}^{n} t_j$</td>
</tr>
<tr>
<td>$c_6$</td>
<td>$\sum_{j=2}^{n} (t_j - 1)$</td>
</tr>
<tr>
<td>$c_7$</td>
<td>$\sum_{j=2}^{n} (t_j - 1)$</td>
</tr>
<tr>
<td>$c_8$</td>
<td>$n - 1$</td>
</tr>
</tbody>
</table>
**INSERTION-SORT**($A$)

1. **for** $j = 2$ **to** $A.length$
2. \hspace{2em} $key = A[j]$
4. \hspace{2em} $i = j - 1$
5. \hspace{2em} **while** $i > 0$ and $A[i] > key$
6. \hspace{4em} $A[i + 1] = A[i]$
7. \hspace{4em} $i = i - 1$
8. \hspace{2em} $A[i + 1] = key$

<table>
<thead>
<tr>
<th>Cost</th>
<th>Times</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$n$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$c_4$</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$c_5$</td>
<td>$\sum_{j=2}^{n} t_j$</td>
</tr>
<tr>
<td>$c_6$</td>
<td>$\sum_{j=2}^{n} (t_j - 1)$</td>
</tr>
<tr>
<td>$c_7$</td>
<td>$\sum_{j=2}^{n} (t_j - 1)$</td>
</tr>
<tr>
<td>$c_8$</td>
<td>$n - 1$</td>
</tr>
</tbody>
</table>

\[
T(n) = c_1 n + c_2 (n - 1) + c_4 (n - 1)
+ c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n - 1)
\]

... as a function of the input size ...


**INSERTION-SORT**\((A)\)

1. \(\textbf{for } j = 2 \text{ to } A.\text{length} \)
2. \(\quad \textbf{key } = A[j] \)
3. \(\quad // \text{ Insert } A[j] \text{ into the sorted } \)
   \(\quad \text{sequence } A[1 \ldots j - 1]. \)
4. \(\quad i = j - 1 \)
5. \(\quad \textbf{while } i > 0 \text{ and } A[i] > \textbf{key} \)
6. \(\quad \quad A[i + 1] = A[i] \)
7. \(\quad i = i - 1 \)
8. \(\quad A[i + 1] = \textbf{key} \)

\[T(n) = c_1 n + c_2(n - 1) + c_4(n - 1) \]
\[+ c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n - 1)\]
**INSERTION-SORT** *(A)*

1. for *j* = 2 to *A*.length
2. \( key = A[j] \)
4. \( i = j - 1 \)
5. while \( i > 0 \) and \( A[i] > key \)
7. \( i = i - 1 \)
8. \( A[i + 1] = key \)

\[
T(n) = c_1 n + c_2 (n - 1) + c_4 (n - 1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8 (n - 1)
\]

**T(n):** Depends on which input of size *n*
... in typical cases.
... in typical cases.

Problem $P$    Algorithm $A$
... in typical cases.

Problem $P$  

Algorithm $A$

Inputs: $\mathcal{X}_n$ of size $n$
... in typical cases.

Problem $P$  Algorithm $A$

Inputs: $\mathcal{X}_n$ of size $n$

\[
W(n) = \max_{x \in \mathcal{X}_n} T(x)
\]
\[ \ldots \text{in typical cases.} \]
... in typical cases.

**Problem** \( P \quad \text{Algorithm} \ A \)

**Inputs:** \( \mathcal{X}_n \) of size \( n \)

\[
W(n) = \max_{x \in \mathcal{X}_n} T(x)
\]

\[
B(n) = \min_{x \in \mathcal{X}_n} T(x)
\]

\[
A(n) = \sum_{x \in \mathcal{X}_n} T(x) \cdot P(x)
\]
... in typical cases.

Problem $P$    Algorithm $A$

Inputs: $\mathcal{X}_n$ of size $n$

$$W(n) = \max_{x \in \mathcal{X}_n} T(x)$$

$$B(n) = \min_{x \in \mathcal{X}_n} T(x)$$

$$A(n) = \sum_{x \in \mathcal{X}_n} T(x) \cdot P(x) = \mathbb{E}[T]$$
... in typical cases.

Problem $P$    Algorithm $A$

Inputs: $\mathcal{X}_n$ of size $n$

$$W(n) = \max_{x \in \mathcal{X}_n} T(x)$$

$$B(n) = \min_{x \in \mathcal{X}_n} T(x)$$

$$A(n) = \sum_{x \in \mathcal{X}_n} T(x) \cdot P(x) = \mathbb{E}[T] = \sum_{t \in T(\mathcal{X}_n)} t \cdot P(T = t)$$
**Insertion-Sort** \( A \)

1. **for** \( j = 2 \) **to** \( A.length \)
2. \( key = A[j] \)
3. // Insert \( A[j] \) into the sorted
   sequence \( A[1..j-1] \).
4. \( i = j - 1 \)
5. **while** \( i > 0 \) and \( A[i] > key \)
7. \( i = i - 1 \)
8. \( A[i + 1] = key \)

<table>
<thead>
<tr>
<th>cost</th>
<th>times</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1 )</td>
<td>( n )</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>( n - 1 )</td>
</tr>
<tr>
<td>0</td>
<td>( n - 1 )</td>
</tr>
<tr>
<td>( c_4 )</td>
<td>( n - 1 )</td>
</tr>
<tr>
<td>( c_5 )</td>
<td>( \sum_{j=2}^{n} t_j )</td>
</tr>
<tr>
<td>( c_6 )</td>
<td>( \sum_{j=2}^{n} (t_j - 1) )</td>
</tr>
<tr>
<td>( c_7 )</td>
<td>( \sum_{j=2}^{n} (t_j - 1) )</td>
</tr>
<tr>
<td>( c_8 )</td>
<td>( n - 1 )</td>
</tr>
</tbody>
</table>
**Insertion-Sort**(\(A\))

```
1     for j = 2 to A.length
2        key = A[j]
3        // Insert A[j] into the sorted sequence A[1..j-1].
4        i = j - 1
5        while i > 0 and A[i] > key
6            A[i+1] = A[i]
7            i = i - 1
8        A[i+1] = key
```

\[
B(n) = (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)
\]
**INSERTION-SORT** *(A)*

1. **for** *j = 2 to A.length*
2. \( key = A[j] \)
3. \( \text{// Insert } A[j] \text{ into the sorted sequence } A[1..j-1]. \)
4. \( i = j - 1 \)
5. **while** *i > 0 and A[i] > key*
7. \( i = i - 1 \)
8. \( A[i+1] = key \)

\[
B(n) = (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)
\]

\[
W(n) = \frac{c_5 + c_6 + c_7}{2}n^2 + \left( c_1 + c_2 + c_4 + c_8 - \frac{c_5 + c_6 + c_7}{2} \right)n - (c_2 + c_4 + c_5 + c_8)
\]
**Insertion-Sort** $(A)$

1. for $j = 2$ to $A.length$
2. \hspace{1cm} key = A[j]
3. \hspace{1cm} // Insert $A[j]$ into the sorted sequence $A[1..j-1]$.
4. \hspace{1cm} $i = j - 1$
5. \hspace{1cm} while $i > 0$ and $A[i] > key$
6. \hspace{1cm} \hspace{1cm} $A[i+1] = A[i]$
7. \hspace{1cm} \hspace{1cm} $i = i - 1$
8. \hspace{1cm} $A[i+1] = key$

\[
B(n) = (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)
\]

\[
W(n) = \frac{c_5 + c_6 + c_7}{2} n^2 + \left( c_1 + c_2 + c_4 + c_8 - \frac{c_5 + c_6 + c_7}{2} \right) n - (c_2 + c_4 + c_5 + c_8)
\]

\[
A(n) =
\]
Insertion-Sort($A$)

1. for $j = 2$ to $A$.length
2. \hspace{1em} key = $A[j]$
4. \hspace{1em} $i = j - 1$
5. \hspace{1em} while $i > 0$ and $A[i] > key$
6. \hspace{2em} $A[i + 1] = A[i]$
7. \hspace{2em} $i = i - 1$
8. \hspace{1em} $A[i + 1] = key$

\[
B(n) = (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)
\]

\[
W(n) = \frac{c_5 + c_6 + c_7}{2}n^2 + (c_1 + c_2 + c_4 + c_8 - \frac{c_5 + c_6 + c_7}{2})n - (c_2 + c_4 + c_5 + c_8)
\]

\[
A(n) = 2.25n^2 + 7.75n - 3H_n - 6 \quad (H_n = \sum_{k=1}^{n} \frac{1}{k} \approx \ln n)
\]
Q: How fast is your algorithm?
Q: How fast is your algorithm?

listen carefully.
Q: How fast is your algorithm?

\[ W(n) = \frac{c_5 + c_6 + c_7}{2} n^2 + \left( c_1 + c_2 + c_4 + c_8 - \frac{c_5 + c_6 + c_7}{2} \right) n - (c_2 + c_4 + c_5 + c_8) \]
BIGOMICRON AND BIG OMEGA AND BIG THETA

Donald E. Knuth
Computer Science Department
Stanford University
Stanford, California 94305

Reference:
“Big Omicron and Big Omega and Big Theta”, Donald E. Knuth, 1976.
BIGOMICRON AND BIG OMEGA AND BIG THETA

Donald E. Knuth
Computer Science Department
Stanford University
Stanford, California 94305

Reference:
“Big Omicron and Big Omega and Big Theta”, Donald E. Knuth, 1976.

Asymptotics
Q: How fast is your algorithm?

\[ W(n) = \frac{c_5 + c_6 + c_7}{2} n^2 + (c_1 + c_2 + c_4 + c_8 - \frac{c_5 + c_6 + c_7}{2}) n - (c_2 + c_4 + c_5 + c_8) \]
$W(n) = \frac{c_5 + c_6 + c_7}{2} n^2 + (c_1 + c_2 + c_4 + c_8 - \frac{c_5 + c_6 + c_7}{2}) n - (c_2 + c_4 + c_5 + c_8)$

$W(n) = O(n^2)$
Q: How fast is your algorithm?

\[ W(n) = \frac{c_5 + c_6 + c_7}{2} n^2 + (c_1 + c_2 + c_4 + c_8 - \frac{c_5 + c_6 + c_7}{2}) n - (c_2 + c_4 + c_5 + c_8) \]

\[ W(n) = O(n^2) \]

“Order at most \( n^2 \)”
Q: How fast is your algorithm?

\[ W(n) = \frac{c_5 + c_6 + c_7}{2} n^2 + (c_1 + c_2 + c_4 + c_8 - \frac{c_5 + c_6 + c_7}{2}) n - (c_2 + c_4 + c_5 + c_8) \]

\[ W(n) = O(n^2) \]

“Order at most \( n^2 \)”

“\( W(n) \) is a function whose order of magnitude is upper-bounded by a constant times \( n^2 \), for all large \( n \).”
\[ f(n) = O(g(n)) \]

\( f(n) \) is a function whose order of magnitude is upper-bounded by a constant times \( g(n) \), for all large \( n \).

\[ O(g(n)) = \{ f(n) | \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \} \]
\[ f(n) = O(g(n)) \]

“\( f(n) \) is a function whose order of magnitude is upper-bounded by a constant times \( g(n) \), for all large \( n \).”
\[ f(n) = O(g(n)) \]

“\( f(n) \) is a function whose order of magnitude is upper-bounded by a constant times \( g(n) \), for all large \( n \).”
\[ f(n) = O(g(n)) \]

"\( f(n) \) is a function whose order of magnitude is upper-bounded by a constant times \( g(n) \), for all large \( n \)."

\[ O(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \right\} \]
\[ f(n) = O(g(n)) \]

\[ O(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \right\} \]
$f(n) = O(g(n))$

$O(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \right\}$
\[ f(n) = O(g(n)) \]

\[
O(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \right\}
\]

It is a tradition to write \( f(n) = O(g(n)) \) instead of \( f(n) \in O(g(n)) \).
\[42n^2 + 2020n = O(n^2)\]
\[42n^2 + 2020n = O(n^2) = O(n^3)\]
$42n^2 + 2020n = O(n^2) = O(n^3)$

$42n^2 + 2020n \in O(n^2) \subseteq O(n^3)$
\[ O(f(n)) + O(g(n)) = \]
\[ O(f(n)) + O(g(n)) \triangleq \{ h + l \mid h \in O(f(n)), l \in O(g(n)) \} \]
\[ O(f(n)) + O(g(n)) \triangleq \{ h + l \mid h \in O(f(n)), l \in O(g(n)) \} \]

\[ O(f(n))O(g(n)) \triangleq \{ hl \mid h \in O(f(n)), l \in O(g(n)) \} \]
\[ O(f(n)) + O(g(n)) \triangleq \left\{ h + l \mid h \in O(f(n)), l \in O(g(n)) \right\} \]

\[ O(f(n))O(g(n)) \triangleq \left\{ hl \mid h \in O(f(n)), l \in O(g(n)) \right\} \]

\[ O(f(n)) - O(g(n)) \triangleq \]
\[ O(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \right\} \]
\[ O(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \right\} \]

\[ 42n = O(0.50n^2) \]
$$O(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \right\}$$

\[ 42n = O(0.50n^2) \quad 42n^2 = O(0.50n^2) \]
\[
O(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \right\}
\]

\[
42n = O(0.50n^2) \quad 42n^2 = O(0.50n^2)
\]

\[Q:\text{What does }O(1)\text{ mean?}\]
\[ O(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \right\} \]

\[ 42n = O(0.50n^2) \quad 42n^2 = O(0.50n^2) \]

**Q**: What does \( O(1) \) mean?

**A**: It means constants.
\[ \Omega(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) \leq f(n) \right\} \]
\[\Omega(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) \leq f(n)\right\}\]

\[0.50n^2 = \Omega(42n)\]
\[\Omega(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) \leq f(n) \right\}\]

\[0.50n^2 = \Omega(42n) \quad 0.50n^2 = \Omega(42n^2)\]
\[ \Omega(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) \leq f(n) \right\} \]

\[ 0.50n^2 = \Omega(42n) \quad 0.50n^2 = \Omega(42n^2) \]

\[ \Theta(g(n)) = \left\{ f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq c_1g(n) \leq f(n) \leq c_2g(n) \right\} \]
\[ \Omega(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) \leq f(n) \right\} \]

\[ 0.50n^2 = \Omega(42n) \quad 0.50n^2 = \Omega(42n^2) \]

\[ \Theta(g(n)) = \left\{ f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq c_1g(n) \leq f(n) \leq c_2g(n) \right\} \]

\[ 0.50n^2 = \Theta(42n^2) \]
\[ o(g(n)) = \left\{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < cg(n) \right\} \]
\[ o(g(n)) = \left\{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < cg(n) \right\} \]

\[ 42n = o(0.50n^2) \]
\[ o(g(n)) = \left\{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < cg(n) \right\} \]

\[ 42n = o(0.50n^2) \]

\[ \omega(g(n)) = \left\{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) < f(n) \right\} \]
\[ o(g(n)) = \left\{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < cg(n) \right\} \]

\[ 42n = o(0.50n^2) \]

\[ \omega(g(n)) = \left\{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) < f(n) \right\} \]

\[ 0.50n^2 = \omega(42n) \]
$O \quad \Omega \quad \Theta$

$o \quad \omega \quad \theta$
\[ O \, \Omega \, \Theta \]
\[ o \, \omega \, \theta \]

\[ f(n) \sim g(n) \iff \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1 \]
\[ f(n) \sim g(n) \iff \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1 \]

\[ 42n^2 + 2020n \sim 42n^2 + 2019n \]
\[ f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \land f(n) = \Omega(g(n)) \]

\[ f(n) = O(g(n)) \iff g(n) = \Omega(f(n)) \]

\[ f(n) = o(g(n)) \iff g(n) = \omega(f(n)) \]
\[ O(f(n)) + O(g(n)) = O(f(n) + g(n)) \]
\[ O(f(n)) + O(g(n)) = O(f(n) + g(n)) \]

\[ O(f(n))O(g(n)) = O(f(n)g(n)) \]
Q: How to compare functions in terms of $O/\Omega/\Theta$?
Q: How to compare functions in terms of $O/\Omega/\Theta$?

\[ O(1) = O(\log \log n) = O(\log n) = O((\log n)^c) = O(n^{\epsilon}) = O(n^c) = O(n^c \log n) = O(n^{\log n}) = O(c^n) = O(n^n) \]

\[ (0 < \epsilon < 1 < c) \]
Stirling Formula (by James Stirling):

\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \]
Stirling Formula (by James Stirling):

\[ n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \]

\[ \log(n!) = \Theta(n \log n) \]
Stirling Formula (by James Stirling):

\[ n! \sim \sqrt{2\pi n \left(\frac{n}{e}\right)^n} \]

\[ \log(n!) = \Theta(n \log n) \]

\[ H_n = \sum_{k=1}^{n} \frac{1}{k} = \Theta(\log n) \]
Learning by Doing
\[ A[0, \ldots n - 1] \quad 1 \leq l \leq n \]
A[0, \ldots n - 1] \quad 1 \leq l \leq n

\text{rotate}(A, n, l) : \text{Rotate } A \text{ left by } l \text{ places}
A[0, \ldots n − 1] \quad 1 \leq l \leq n

\text{rotate}(A, n, l) : \text{Rotate } A \text{ left by } l \text{ places}

\begin{array}{ccccc}
0 & 1 & 2 & 3 & 4 \\
\end{array}

\begin{array}{ccccc}
3 & 4 & 0 & 1 & 2 \\
\end{array}
\[ A[0, \ldots, n - 1] \quad 1 \leq l \leq n \]

\text{ROTATE}(A, n, l): \text{Rotate } A \text{ left by } l \text{ places}

Critical Operation: copy
1: **procedure** ROTATE($A, n, l$)
2:    **for** $i = 1 \ldots l$ **do**
3:       ROTATE-BY-ONE($A, n$)
1: **procedure** ROTATE($A, n, l$)
2: for $i = 1 \ldots l$ do
3: ROTATE-BY-ONE($A, n$)
1: procedure ROTATE($A, n, l$)
2: for $i = 1 \ldots l$ do
3: \hspace{1em} ROTATE-BY-ONE($A, n$)

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Time</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
1: procedure ROTATE(A, n, l)
2:   for i = 1 \ldots l do
3:     ROTATE-BY-ONE(A, n)

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Time</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>rotate-one-by-one</td>
<td>(nl = O(n^2))</td>
<td>(O(1))</td>
</tr>
</tbody>
</table>
1: **procedure** ROTATE\( (A, n, l) \)
2: copy \( A[0 \ldots l - 1] \) into \( v \)
3: move \( A[l \ldots n - 1] \) left \( l \) places
4: copy \( v \) to \( A[l \ldots n - 1] \)
1: **procedure** ROTATE(A, n, l)
2: \hspace{1em} copy A[0...l − 1] into v
3: \hspace{1em} move A[l...n − 1] left l places
4: \hspace{1em} copy v to A[l...n − 1]
1: **procedure** ROTATE(A, n, l)
2: copy A[0...l − 1] into v
3: move A[l...n − 1] left l places
4: copy v to A[l...n − 1]
1: procedure ROTATE(A, n, l)
2: copy A[0...l - 1] into v
3: move A[l...n - 1] left l places
4: copy v to A[l...n - 1]

Algorithm | Time  | Space |
-----------|-------|-------|
rotate-copy | $O(n)$ | $l = O(n)$
$n = 5, \quad l = 3$

\begin{tabular}{c c c c c c}
0 & 1 & 2 & 3 & 4 \\
3 & 4 & 0 & 1 & 2 \\
\end{tabular}
\[ n = 5, \quad l = 3 \]

\[ \begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
\end{array} \]

\[ \begin{array}{cccccc}
3 & 4 & 0 & 1 & 2 \\
\end{array} \]

\((0, 2, 4, 1, 3)\)
\( n = 5, \quad l = 3 \)

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
3 & 4 & 0 & 1 & 2 \\
\end{array}
\]

\((0, 2, 4, 1, 3)\)

\( n = 9, \quad l = 6 \)

\[
\begin{array}{ccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

\((0, 3, 6)\) \quad \((1, 7, 4)\) \quad \((2, 8, 5)\)
Correctness Proof?
Correctness Proof?

Permutations as **Product** of Disjoint Cycles
Correctness Proof?

Permutations as **Product** of Disjoint Cycles

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Time</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>rotate-cyclic</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>
Correctness Proof?

Permutations as Product of Disjoint Cycles

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Time</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>rotate-cyclic</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>
\[ B \cdot A = (A^R \cdot B^R)^R \]
\[ B \cdot A = (A^R \cdot B^R)^R \]

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 & 4 \\
2 & 1 & 0 & 4 & 3 \\
3 & 4 & 0 & 1 & 2 \\
\end{array}
\]

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Time</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>rotate-reverse</td>
<td>(O(n))</td>
<td>(O(1))</td>
</tr>
<tr>
<td>Algorithm</td>
<td>Time</td>
<td>Space</td>
</tr>
<tr>
<td>------------------------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>rotate-one-by-one</td>
<td>$O(n^2)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>rotate-copy</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>rotate-cyclic</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>rotate-reverse</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>
Chapter 9: Asymptotics

\[O \quad \Omega \quad \Theta\]

\[o \quad \omega\]
Thank You!
Your opinion Matters

Office 302
Mailbox: H016
hfwei@nju.edu.cn