证明：旅行商问题是NP完全问题

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字丑求不嫌弃，看不懂可以问我
或者直接看图

TSP ∈ NPC

What is TSP?

求带权无向完全图中权值最小的哈密顿回路

转化为 decision problem：是否存在权值≤k的哈密顿回路

$\langle G, c, k\rangle$

证明 L ∈ NPC 的流程

- Step1. $L \in NP$
- ☆Step2 Find L', $L' \le_p L$ (Reduction function)
- Step3. Correctness proof.
- Step4. Poly-time.

clrs's proof

Step1. $TSP \in NP$ (So Easy)

Cert.: $s = v_1, v_2, \cdots = v_n = t$

All $vis[v_i] = True$. && $\sum C(v_i, v_{i+1}) \le k$

$O(n)$.

Step2. $HC \le_p TSP$

$G=(V,E) \longrightarrow (G'=(V,E'), c, k)$

$c(i,j) = \begin{cases} 0 & (i,j) \in E \\ 1 & (i,j) \notin E \end{cases}$

Step3. $G \in HC$ iff. $\langle G', c, 0\rangle \in TSP$.

G存在一条HAM-CYCLE ⟨G', c⟩存在权值为0的回路

"⇒" If $G \in HC$, G contains a HAM-CYCLE h: $s = v_1, v_2, \cdots, v_n = t$

$\Rightarrow \forall e \in (u,v) \in h$, $e \in E \Rightarrow c(u,v) = 0$

$\Rightarrow \sum C(v_i, v_{i+1}) = 0 \Rightarrow \langle G', c, 0\rangle \in TSP$.

"⇐" If $\langle G', c, 0\rangle \in TSP$, $\exists h'$ $c(h') \le 0$.

$\Rightarrow \forall e' = (u', v') \in h'$, $c(u', v') = 0$

$\Rightarrow (u', v') \in E$. $\Rightarrow h' \in HC$

Step4. 两两连边 $O(E) = O(|V|^2)$

更为完整的证明链

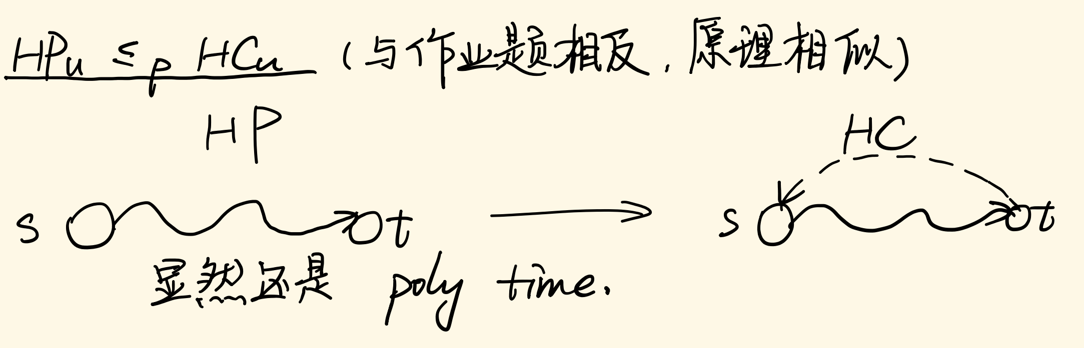
clrs: $\underset{(\text{万恶之源})}{\text{CIRCUIT-SAT}} \le_p SAT \le_p 3\text{-}CNF \le_p CLIQUE \le_p HC \le_p TSP$ (牵扯到的问题太多了)

魏老师给的另一种：

$3\text{-}CNF \le_p HP_{(directed)} \le_p HP_{(undirected)} \le_p HC_{(undirected)} \le_p TSP$

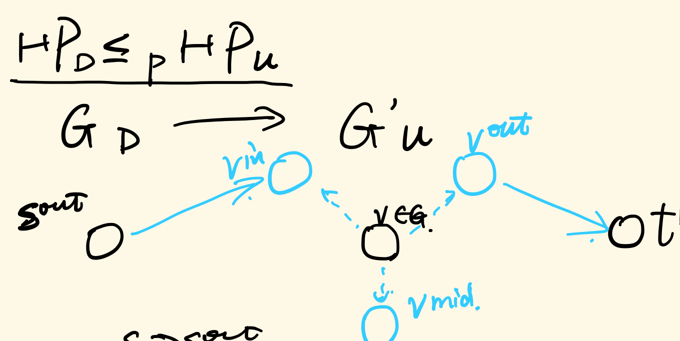
步骤一样，但相对简单清晰

$HP_u \le_p HC_u$ (与作业题相反，原理相似)



显然这是 poly time.

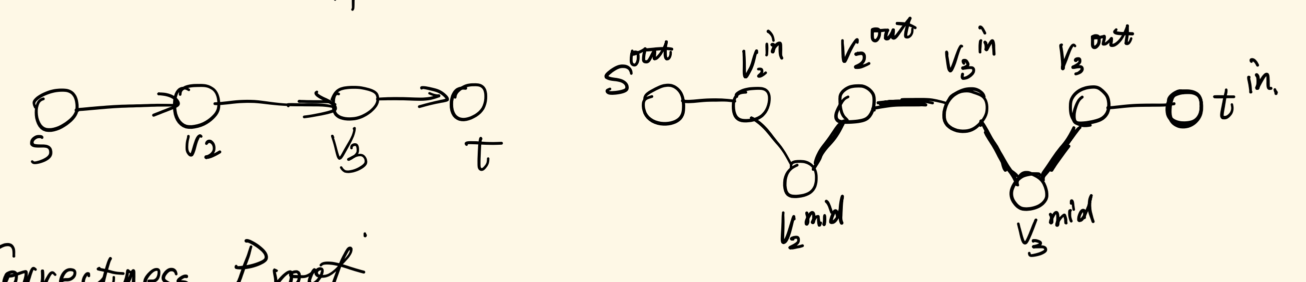
$HP_D \le_p HP_u$



点：$\begin{cases} s \to s^{out} \\ t \to t^{in} \\ v \to v^{in}, v^{out}, v^{mid} \end{cases}$

有点网络流建图的感觉

边：$(u,v) \in E$ iff $(u^{out}, v^{in}) \in E'$



Correctness Proof.

"⇒" If $\exists HP_D \in G$ $s = v_1, v_2, \cdots, v_n = t$.

G': $s^{out}, v_2^{in}, v_2^{mid}, v_2^{out}, v_3^{in}, \cdots, t^{in}$. (HP_u)

"⇐" If $\exists HP_u \in G'$

Claim: $\forall HP_u (s^{out} \rightsquigarrow t^{in})$, HP_u must go from a triple of nodes to another.

$s^{out} \to v_i^{in}$

$v_i^{in} \to v_i^{mid} \to v_i^{out}$

$v_i^{out} \to t^{in}$.

G' 是一种具有特定结构的图
但已经足够证明
(详见中文版clrs P640 上半)

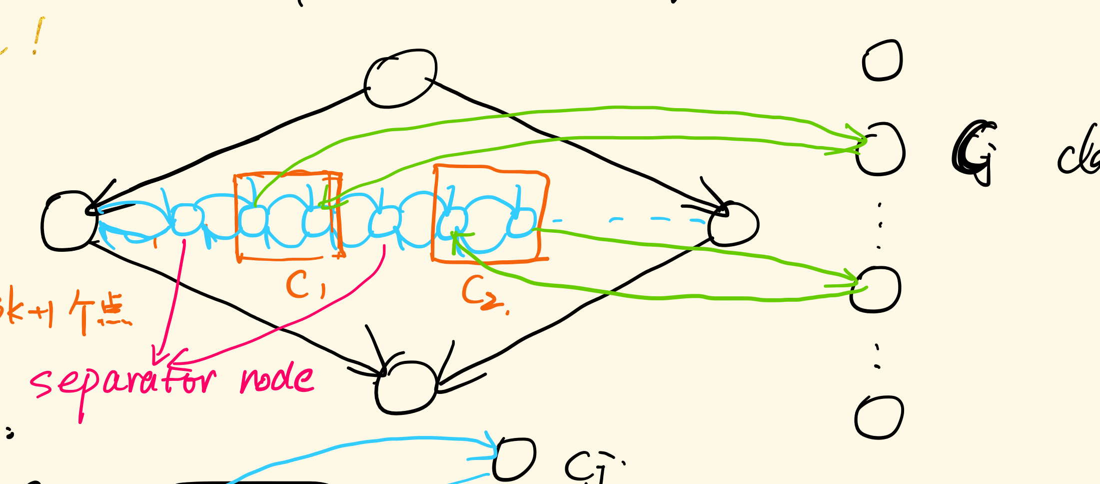
Poly-time: $\underset{\text{建点}}{O(|V|)} + \underset{\text{建边}}{O(|E|)}$

$3\text{-}CNF \le_p HP_D$ (*)

3-CNF: $\phi = (\alpha_1 \vee \beta_1 \vee \gamma_1) \wedge (\alpha_2 \vee \beta_2 \vee \gamma_2) \wedge \cdots \wedge (\alpha_k \vee \beta_k \vee \gamma_k)$

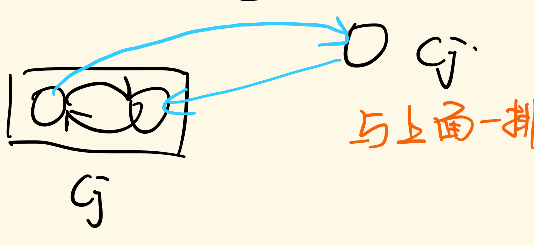
G_D Magic!

variable x_i $(\alpha_i / \beta_i / \gamma_i)$

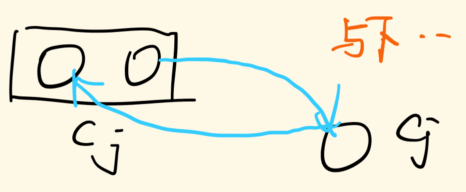


x_i 与 C_j 的关系：

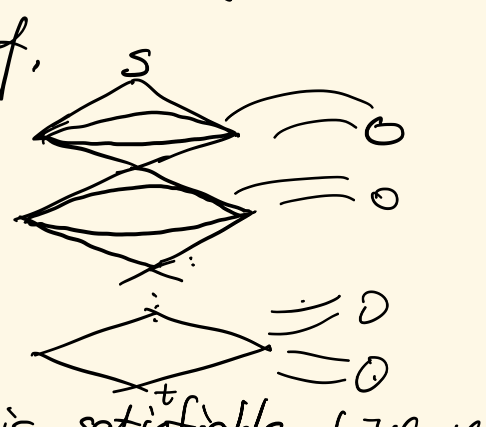
If $x_i \in C_j$



If $\bar{x}_i \in C_j$



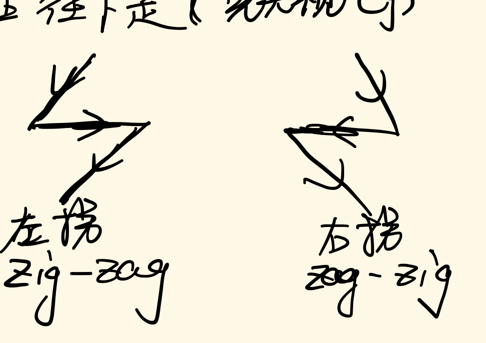
Correctness Proof.



"⇒" If ϕ is satisfiable. $(\exists \varphi, \varphi(\phi) = T)$

$G: s \to t$. 肯定是迂回着从上往下走(先无视 C_j)

有2种迂回方式.



考虑 C_j，从每个clause中挑出一个赋值为真的variable.
(每个clause里至少有1个)

If choose $x_i \in C_j$ 走 代替 (左拐)

If choose $\bar{x}_i \in C_j$ 走 代替 (右拐)

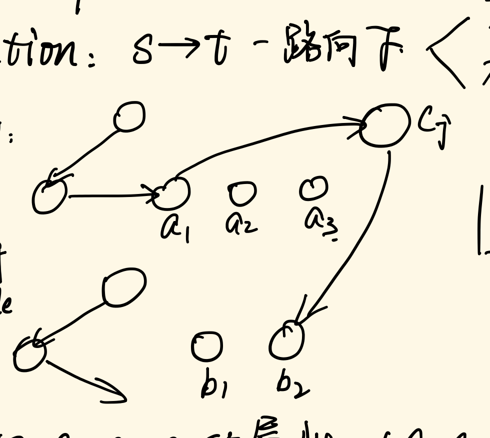
可以保证：每个 C_j 都可以且仅被访问一次

"⇐" If G has $HP_u (s \rightsquigarrow t)$

Best situation: $s \to t$ 一路向下 $\begin{cases} \text{左拐 } \varphi(x_i) = T \\ \text{右拐 } \varphi(x_i) = F \end{cases}$

会不会出现：

Cj连着
两个不属于
同个variable
的结点



Impossible.

Pf：分析 a_1, a_2, a_3 的属性 (a_2, a_3 至少有1个separator node)

Case 1: a_2 是 sep……

$\Rightarrow$ 只有 a_1, a_3 有进入 a_2 的边 $\begin{cases} \text{走} a_1 \to a_2 \times & \because a_1 \text{已经走过了} (a_1 \to C_j) \\ \text{走} a_3 \to a_2 \times & \because \text{即使} a_3 \to a_2, a_2 \text{只能走到} a_3, \text{导致} a_3 \text{被二次访问} \end{cases}$

Case 2: a_3 是 sep……

$\Rightarrow$ 只有 a_1, a_3, C_j 有进入 a_2 的边 $\begin{cases} \text{走} a_1 \to a_2 \times & \text{原因同上} \\ \text{走} a_3 \to a_2 \times & \text{——同上} \\ \text{走} C_j \to a_2 & \text{原因同} a_1 \end{cases}$

Reduction Time.

建点 $O(\underset{\#var}{3k}) O(\underset{\text{in per 菱形}}{3k+1}) + O(\underset{\#clause}{k})$

建边 $O(\underset{\text{菱形}}{2(3k+2)+4}) \cdot O(\underset{\#var}{3k}) + O(\underset{\#clause}{2k})$

} poly-time