反馈与讨论

2014-5-8

5. *Principle of inclusion and exclusion for probability.* The probability of the union $E_1 \cup E_2 \cup \cdots \cup E_n$ of events in a sample space S is given by

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k: \\ 1 \le i_1 < i_2 < \cdots < i_k \le n}} P\left(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}\right).$$

6. *Hatcheck problem.* The *hatcheck problem,* or *derangement problem,* asks for the probability that a bijection of an n-element set maps no element to itself. The answer is

$$\sum_{i=2}^{n}(-1)^i\frac{1}{i!}=\frac{1}{2}-\frac{1}{3!}+\cdots+\frac{(-1)^n}{n!},$$

which is the result of truncating the power series expansion of e^{-1} at the $(-1)^n/n!$ term. Thus, the result is very close to $1/e$, even for relatively small values of n.

7. *Principle of inclusion and exclusion for counting.*

$$\left|\bigcup_{i=1}^{n} E_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k: \\ 1 \le i_1 < i_2 < \cdots < i_k \le n}} \left|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}\right|.$$

2. The eight kings and queens are removed from a deck of 52 cards, and then two of these cards are selected. What is the probability that the king or queen of spades is among the cards selected?

9. The boss asks the secretary to stuff n letters into envelopes, forgetting to mention that he has been adding notes to the letters and, in the process, has rearranged the letters but not the envelopes. In how many ways can the letters be stuffed into the envelopes so that nobody gets the letter intended for him or her? What is the

$$\left|\bigcup_{i=1}^{n} E_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{i_1, i_2, \ldots, i_k: \\ 1 \le i_1 < i_2 < \cdots < i_k \le n}} \left|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}\right|.$$

Let Ei stands for the event that person i gets the right letter.

The result is n!-|UEi| = .

9. $\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)! = \sum_{i=0}^{n}(-1)^i\frac{n!}{i!} = \sum_{i=2}^{n}(-1)^i\frac{n!}{i!}$; $\sum_{i=0}^{n}\frac{(-1)^i}{i!} = \sum_{i=2}^{n}\frac{(-1)^i}{i!}$

10. If you are hashing n keys into a hash table with k locations, what is the probability that every location gets at least one key?

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$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n\frac{1}{k^n}$$

14. A group of n married couples sits around a circular table for a discussion of marital problems. The counselor assigns each person to a seat at random. What is the probability that no husband and wife are side-by-side?

14. $\sum_{k=0}^{n}(-1)^k 2^k \binom{n}{k} \frac{(2n-k-1)!}{(2n-1)!}$

15. Suppose you have a collection of m objects and a set P of p "properties." (We won't define the term "property," but note that a property is something the objects may or may not have.) For each subset S of the set P of all properties, define $N_a(S)$ to be the number of objects in the collection that have *at least* the properties in S (a is for "at least"). Thus, for example, $N_a(\emptyset) = m$. In a typical application, formulas for $N_a(S)$ for other sets $S \subseteq P$ are not difficult to figure out. Define $N_e(S)$ to be the number of objects in our collection that have *exactly* the properties in S (e is for "exactly"). Show that

$$N_e(\emptyset) = \sum_{K:K \subseteq P} (-1)^{|K|} N_a(K).$$

Explain how this formula could be used to compute the number of onto functions in a more direct way than we did when using unions of sets. How would this formula apply to Problem 9?

Important Concepts, Formulas, and Theorems

1. *Conditional probability*. The *conditional probability* of E given F, denoted by $P(E|F)$ and read as "the probability of E given F," is defined by

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

when $P(F) \neq 0$.

2. *Bayes' Theorem*. The relationship between $P(E|F)$ and $P(F|E)$ is

$$P(E|F) = \frac{P(F|E)P(E)}{P(F)}.$$

3. *Independent*. We say E is *independent* of F if $P(E|F) = P(E)$.

4. *Product principle for independent probabilities*. The product principle for independent probabilities (Theorem 5.5) gives another test for independence. Suppose E and F are events in a sample space. Then E is independent of F if and only if $P(E \cap F) = P(E)P(F)$.

5. *Symmetry of independence*. The event E is independent of the event F if and only if F is independent of E.

6. *Independent trials process.* A process that occurs in stages is called an *independent trials process* if, for each sequence $a_1, a_2, \ldots, a_n$ with $a_i \in S_i$,

$$P(x_i = a_i | x_1 = a_1, \ldots, x_{i-1} = a_{i-1}) = P(x_i = a_i).$$

7. *Probabilities of outcomes in independent trials.* In an independent trials process, the probability of a sequence $a_1, a_2, \ldots, a_n$ of outcomes is $P(\{a_1\})P(\{a_2\}) \cdots P(\{a_n\})$.
8. *Coin flipping.* Repeatedly flipping a coin is an independent trials process.
9. *Hashing.* Hashing a list of n keys into k slots is an independent trials process with n stages.
10. *Tree diagram.* In a tree diagram for a multistage process, each level of the tree corresponds to one stage of the process. Each vertex is labeled with one of the possible outcomes at the stage it represents. Each edge is labeled with a conditional probability—the probability of getting the outcome at its right end given the sequence of outcomes that have occurred so far. Each path from the root to a leaf represents a sequence of outcomes and is labeled with the product of the probabilities along that path. This is the probability of that sequence of outcomes.

3. In three flips of a coin, is the event of getting at most one tail independent of the event that not all flips are identical?

4. What is the sample space that you use for rolling two dice, a first one and then a second one? Using this sample space, explain why the event "i dots are on top of the first die" and the event "j dots are on top of the second die" are independent if you roll two dice.