计算机问题求解 – 论题2-4 - 组合与计数

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However, the sum principle will prove to be useful in a variety of problems. Thus, the value of abstraction is that recognizing the abstract elements of a problem often helps us solve subsequent problems.

问题1：

计数在算法分析中为什么很重要？

```
(1)  for i = 1 to n − 1
(2)      for j = i + 1 to n
(3)          if (A[i] > A[j])
(4)              exchange A[i] and A[j]
```

How many times is the comparison $A[i] > A[j]$ made in Line 3?

critical operation

Principle 1.1 (Sum Principle)

The size of a union of a family of mutually disjoint finite sets is the sum of the sizes of the sets.

$$\left|\bigcup_{i=1}^{m} S_i\right| = \sum_{i=1}^{m} |S_i|.$$

问题2：

你如何理解这里所体现的“抽象”过程？

我们究竟要数什么？

你能解释一下抽象的过程吗？

```
(1)  for i = 1 to r
(2)      for j = 1 to m
(3)          S = 0
(4)          for k = 1 to n
(5)              S = S + A[i,k] * B[k,j]
(6)          C[i,j] = S
```

How many multiplications (expressed in terms of r, m, and n) does this pseudocode carry out in total among all the iterations of Line 5?

$$T_i = \bigcup_{j=1}^{m} S_j. \qquad |T_i| = \left|\bigcup_{j=1}^{m} S_j\right| = \sum_{j=1}^{m} |S_j| = \sum_{j=1}^{m} n = mn.$$

操作计数与子集计数

相同的情况，不同的抽象：

在排序的例子里，对任意含两个元素的子集，我们做一次比较，则比较次数等于n个元素的集合所有的两个元素的子集的个数。

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

```
for i = 1 to n − 1
    minval = A[i]
    minindex = i
    for j = i to n
        if (A[j] < minval)
            minval = A[j]
            minindex = j
    exchange A[i] and A[minindex]
bigjump = 0
for i = 2 to n
    if (A[i] > 2 * A[i − 1])
        bigjump = bigjump + 1
```

easily solvable pieces. If we can decompose the problem into smaller pieces and solve the smaller pieces, then we may be able to either add or multiply solutions to smaller problems in order to solve the larger problem. In this

乘法原则的两个版本不一样吗？

问题3：通俗地说说这是什么意思？

Principle 1.4 (Product Principle, Version 2)

If a set S of lists of length m has the properties that

1. there are i_1 different first elements of lists in S, and
2. for each $j > 1$ and each choice of the first $j - 1$ elements of a list in S, there are i_j choices of elements in position j of those lists,

then there are $i_1 i_2 \cdots i_m = \prod_{k=1}^{m} i_k$ lists in S.

乘法原则的两个版本不一样吗？

Principle 1.3 (Product Principle)

The size of a union of m disjoint sets, each of size n, is mn.

Principle 1.4 (Product Principle, Version 2)

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从数list到数函数

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问题4：这与数函数有什么关系？

问题6：

你能解释这个原理的应用吗？

```
(1)  trianglecount = 0
(2)  for i = 1 to n
(3)      for j = i+1 to n
(4)          for k = j+1 to n
(5)              if points i, j, and k are not collinear
(6)                  trianglecount = trianglecount + 1
```

Among all iterations of line 5 of the pseudocode, what is the total number of times this line checks three points to see if they are collinear?

Principle 1.5 **(Bijection Principle)**

Two sets have the same size if and only if there is a one-to-one function from one set onto the other.

在插入排序分析中使用双射原理

我们用数“逆序”的个数来代替数比较操作的个数。

我们还可以对“行程”计数：

如果 $a_1, a_2, \ldots a_n$ 是 $\{1,2,\ldots,n\}$ 元素的任意一种排列, 要想转变成递增序，元素必须移动的“总行程”是：

$$|a_1-1|+|a_2-2|+\ldots+|a_n-n|$$

For a spcific $j\,(1\le j\le n)$, *the average value of* $|a_j - j|$ *is*

$$\frac{1}{n}(|1-j|+|2-j|+\ldots+|n-j|)=\frac{1}{n}\left(\sum_{i=1}^{j-1}(j-i)+\sum_{i=j+1}^{n}(i-j)\right)=\frac{1}{n}\left(\sum_{i=1}^{j-1} i+\sum_{i=1}^{n-j} i\right)$$

sum on j gives $\frac{1}{3}(n^2-1)$

$$\sum_{j=1}^{n}\frac{1}{n}\left(\sum_{i=1}^{j-1} i+\sum_{i=1}^{n-j} i\right)=\frac{2}{n}\sum_{j}^{n}(1+2+\ldots+(j-1))$$

$$=\frac{1}{n}\sum_{j}^{n}(j^2-j)=\frac{1}{6}(n+1)(2n+1)-\frac{1}{2}(n+1)$$

问题7：

你还记得什么是等价关系吗？它和集合分划有什么关系？

等价关系用于计数

问题：

用2种颜色给5个对象着色，并保证每种颜色最少用于2个对象，有多少种不同的着色法?

5个对象不同的排列共有5！种，其中有多少种恰好对应于同一种着色方案?

$\{A, B, C, D, E\}$. Consider the particular labeling in which A, B, and D are labeled blue and C and E are labeled red. Which lists correspond to this labeling? They are

ABDCE	ABDEC	ADBCE	ADBEC	BADCE	BADEC
BDACE	BDAEC	DABCE	DABEC	DBACE	DBAEC,

$$q \cdot 12 = 120,$$

注意: $q = \binom{5}{3} = 10$

给出一个等价类

{A, B, C, D, E}. Consider the particular labeling in which A, B, and D are labeled blue and C and E are labeled red. Which lists correspond to this labeling? They are

$$\begin{array}{cccccc} ABDCE & ABDEC & ADBCE & ADBEC & BADCE & BADEC \\ BDACE & BDAEC & DABCE & DABEC & DBACE & DBAEC, \end{array}$$

在5!个排列构成的集合上，可以考虑定义什么样的等价关系，使得等价类的个数就是不同的着色方案的个数？
在5个元素构成的集合上，有多少个等价关系使得集合划分为2元素和3元素子集？等价关系个数就是着色方案个数

在这里对称因素起什么作用？

等价关系和等价类
在计数中作用巨大

书架上的排列问题

We have k books to arrange on the n shelves of a bookcase. The order in which the books appear on a shelf matters, and each shelf can hold all the books. We will assume that as the books are placed on the shelves, they are pushed as far to the left as they will go. Thus, all that matters is the order in which the books appear. When book i is placed on a shelf, it can go between two books already there or to the left or right of all the books on that shelf.

$$n(n+1)(n+2)\cdots(n+k-1) = \prod_{i=1}^{k}(n+i-1)$$

$$n^{\overline{k}}$$

$$= \prod_{j=0}^{k-1}(n+j) = \frac{(n+k-1)!}{(n-1)!}.$$

问题8：

你能否利用等价关系的概念来解释上述结果中的商式？

$$\frac{(n+k-1)!}{(n-1)!}$$

回想一下书上如何讨论二项式定理的证明。

Multiset问题

The number of k-element multisets chosen from an n-element set is

$$\frac{n^{\overline{k}}}{k!} = \binom{n+k-1}{k}.$$

问题9:
这如何与书架排列问题联系在一起?

define two bookcase arrangements of k books on n shelves to be equivalent if we get one from the other by permuting the books among themselves.

问题10：

为什么我们在算法分析中经常用到"Σ"？

累加经常不简单！

试试看

计算： $\sum_{i=1}^{k} i2^i$

For your reference : Arithmetic - Geometric Series

$$\sum_{i=1}^{k} i2^i = \sum_{i=1}^{k} i\left(2^{i+1} - 2^i\right) \quad \Sigma$$

$$= (2^2 + 2\cdot 2^3 + 3\cdot 2^4 + \cdots + (k-1)\cdot 2^k + k\cdot 2^{k+1})$$

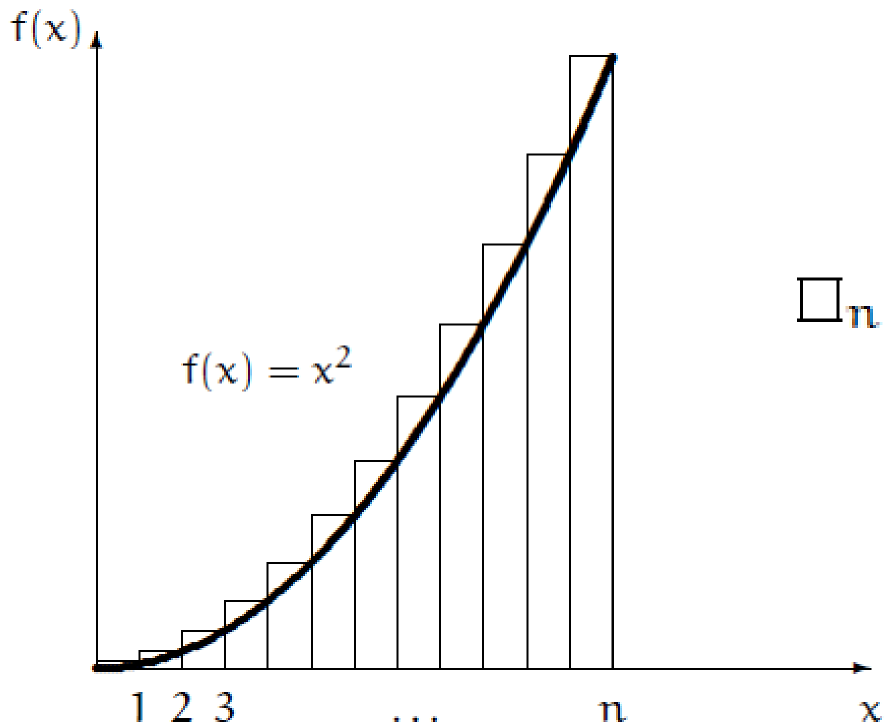
$$- (2 + 2\cdot 2^2 + 3\cdot 2^3 + \cdots + (k-1)\cdot 2^{(k-1)} + k\cdot 2^k)$$

$$= (k\cdot 2^{k+1} - 2) - \sum_{i=2}^{k} 2^i = (k\cdot 2^{k+1} - 2) - (2^{k+1} - 4)$$

$$= (k-1)\cdot 2^{k+1} + 2$$

你还记得吗？

$$\Box_n = \sum_{0 \leqslant k \leqslant n} k^2, \qquad \text{for } n \geqslant 0$$



$$\Box_n = \frac{n(n+1)(2n+1)}{6}$$

The area under this curve is $\int_0^n x^2\,dx = n^3/3$; therefore we know that $\Box_n$ is approximately $\frac{1}{3}n^3$.

有时候需要“生成”

ALGORITHM 1 Generating the Next Permutation in Lexicographic Order.

procedure *next permutation*($a_1 a_2 \cdots a_n$: permutation of
$\{1, 2, \ldots, n\}$ not equal to $n\ \ n-1\ \ \cdots\ \ 2\ \ 1$)
$j := n - 1$
while $a_j > a_{j+1}$
  $j := j - 1$
$\{j$ is the largest subscript with $a_j < a_{j+1}\}$
$k := n$
while $a_j > a_k$
  $k := k - 1$
$\{a_k$ is the smallest integer greater than a_j to the right of $a_j\}$
interchange a_j and a_k
$r := n$
$s := j + 1$
while $r > s$
begin
  interchange a_r and a_s
  $r := r - 1$
  $s := s + 1$
end
{this puts the tail end of the permutation after the jth position in increasing order}

课外作业

- CS pp.8-: 9, 13
- CS pp.20-: 15
- CS pp.30-: 6, 9, 14
- CS pp.54-: 8, 10, 15