计算机问题求解 – 论题2-6

- 概率分析与随机算法

2019年04月01日

Part I
随机变量与算法分析

问题1：

以连续掷10次硬币，数正面出现多少次为例，说明什么是随机变量？为什么说它是函数？与概率是什么关系？

这里*S*是什么？

A random variable for an experiment with a sample space *S* is a *function* that *assigns a number* to each element of *S*. Typically, instead of using *f* to stand for such a function, we use *X*. (At first, a random variable was conceived of as a variable related to an experiment, explaining the use of *X*, but it is very helpful in understanding the mathematics to realize that *X* is actually a function on the sample space.)

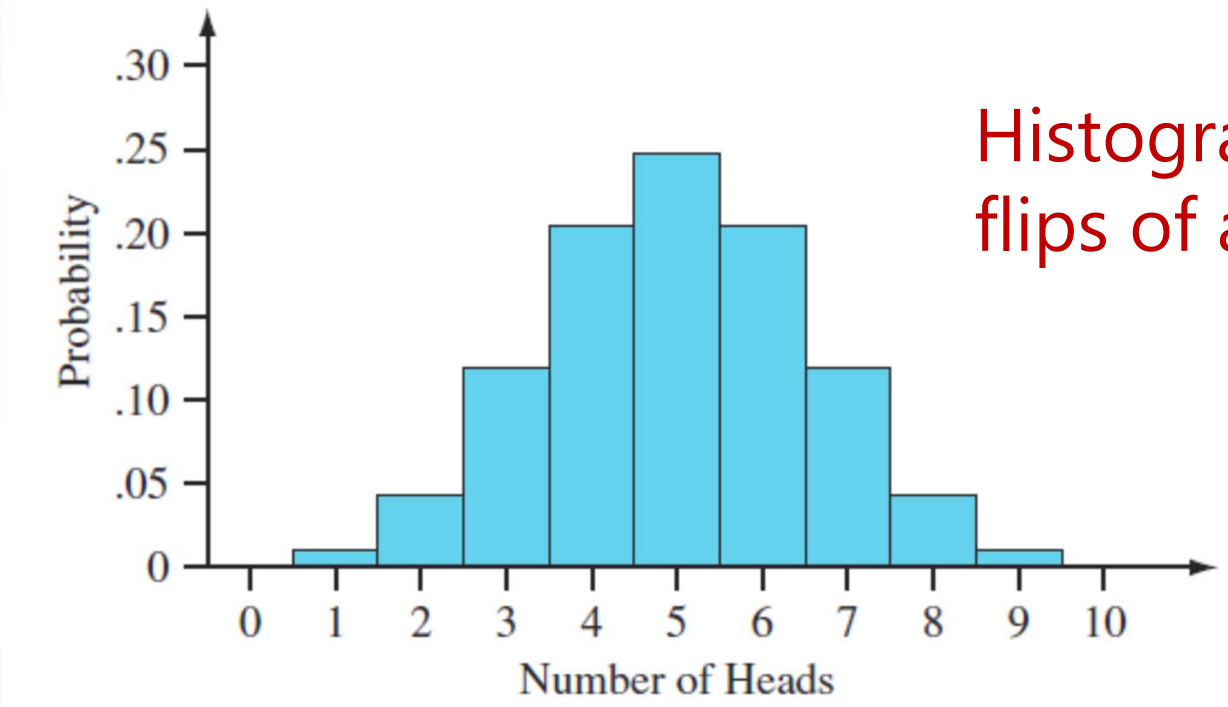
Bernoulli Trials Process

Because the probability of a sequence of outcomes is the product of the probabilities of the individual outcomes, the probability of any sequence of three successes and two failures is $p^3(1-p)^2$. More generally, in n Bernoulli trials, the probability of a given sequence of k successes and $n-k$ failures is $p^k(1-p)^{n-k}$. However, this is not the probability of having k successes, because many different sequences could have k successes.

问题2：

什么意思？

$$P(\text{exactly } k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}.$$



Histogram for 10 flips of a coin

再顺便问一句：“期望值”是出现可能性最大的值吗？

问题3：

你能否结合这个图解释与随机变量相关的以下的概念：期望值、分布以及它们之间的关系？

顺便问一句，纵坐标值怎么得到的？

```
HIRE-ASSISTANT(n)
best = 0                // candidate 0 is a least-qualified dummy candidate
for i = 1 to n
    interview candidate i
    if candidate i is better than candidate best
        best = i
        hire candidate i
```

问题4：

这个算法是“确定”的吗？
什么是“随机”的呢？

Average-Case Analysis of Algorithms

We focus on computing the running time of various algorithms. When the running time of an algorithm is different for different inputs of the same size, we can think of the running time of the algorithm as a random variable on the sample space of inputs, and thus, we can analyze the expected running time of the algorithm. This gives us an understanding different from studying just the worst-case running time for an input of a given size.

Expected running time vs.
the worst-case running time

问题5：

你能否描述一下对应于**hiring problem**的问题空间以及分析算法所需要的随机变量？

Worst-case是什么？需执行多少次hiring？

Indicator Random Variable

$$I\{A\} = \begin{cases} 1 & \text{if } A \text{ occurs ,} \\ 0 & \text{if } A \text{ does not occur} \end{cases}$$

Indicator random variables provide a convenient method for converting between probabilities and expectations.

问题6：

你能否以抛硬币为例解释这句话的意思？

Given a sample space S and an event A in the sample space S, let $X_A = \mathrm{I}\{A\}$. Then $\mathrm{E}[X_A] = \Pr\{A\}$.

$$\begin{aligned} \mathrm{E}[X_A] &= \mathrm{E}[\mathrm{I}\{A\}] \\ &= 1 \cdot \Pr\{A\} + 0 \cdot \Pr\{\overline{A}\} \\ &= \Pr\{A\} , \end{aligned}$$

where $\overline{A}$ denotes $S - A$, the complement of A.

只抛一次硬币体现不了$I\{A\}$的价值，但是：

$$\mathrm{E}[X] = \mathrm{E}\left[\sum_{i=1}^{n} X_i\right]$$

Hiring-Assistant算法的平均情况分析

涉及的随机变量：

- Hiring操作执行次数：*X*；
- 事件"第*i*个候选人被雇用"的indicator：X_i；

$$X = X_1 + X_2 + \cdots + X_n$$

$$\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}\left[\sum_{i=1}^{n} X_i\right] && \text{(by equation (5.2))} \\
&= \sum_{i=1}^{n} \mathrm{E}[X_i] && \text{(by linearity of expectation)} \\
&= \sum_{i=1}^{n} 1/i && \text{(by equation (5.3))} \\
&= \ln n + O(1) && \text{(by equation (A.7))} .
\end{aligned}$$

$$\mathrm{E}[X_i] = 1/i$$

为什么？

Part II
随机算法

随机算法：让“随机”更“随机”

```
RANDOMIZED-HIRE-ASSISTANT(n)
randomly permute the list of candidates
best = 0            // candidate 0 is a least-qualified dummy candidate
for i = 1 to n
    interview candidate i
    if candidate i is better than candidate best
        best = i
        hire candidate i
```

相当于用抛硬币或者掷色子的方式决定下一步该干什么！

问题7：
这个算法还有“最坏情况”吗？

生成序列的“随机”排列

PERMUTE-BY-SORTING(A)

```
n = A.length
let P[1..n] be a new array
for i = 1 to n
        P[i] = RANDOM(1, n^3)
sort A, using P as sort keys
```

$O(n\log n)$

RANDOMIZE-IN-PLACE(A)

```
n = A.length
for i = 1 to n
        swap A[i] with A[RANDOM(i, n)]
```

$O(n)$

对两个算法都必须回答同样的问题：

是否得到任意可能的排列的机会是一样的(uniformly distribution)?

相交事件的概率

对一个特别的排列，证明其概率是1/*n*!：第*i*个元素“权”恰好是第*i*个最小。

用E_i表示对特定的*i*上述条件成立的事件，则要求的排列生成的概率是：

$$\Pr\{E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_{n-1} \cap E_n\}$$

按照条件概率的定义式，并利用归纳法加以推广，上面的式子可以写为：

$$\Pr\{E_1\} \cdot \Pr\{E_2 \mid E_1\} \cdot \Pr\{E_3 \mid E_2 \cap E_1\} \cdot \Pr\{E_4 \mid E_3 \cap E_2 \cap E_1\} \cdots \Pr\{E_i \mid E_{i-1} \cap E_{i-2} \cap \cdots \cap E_1\} \cdots \Pr\{E_n \mid E_{n-1} \cap \cdots \cap E_1\}$$

于是：

$$\Pr\{E_1 \cap E_2 \cap E_3 \cap \cdots \cap E_{n-1} \cap E_n\} = \left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right)\cdots\left(\frac{1}{2}\right)\left(\frac{1}{1}\right) = \frac{1}{n!},$$

只要对上述的“最小”加以不同的解释，这个证明适用于任意排列！

在by-sorting算法中依次将*n*个随机生成的“权”赋给各元素。

利用循环不变式的归纳

针对 In-place 算法 可定义如下的不变式：

Just prior to the ith iteration of the **for** loop of lines 2–3, for each possible $(i-1)$-permutation of the n elements, the subarray $A[1..i-1]$ contains this $(i-1)$-permutation with probability $(n-i+1)!/n!$.

考虑一个特定排列：$\langle x_1, x_2, \ldots, x_i \rangle$

如果第 i 次循环开始前，前i -1个元素恰好是$\langle x_1, x_2, \ldots x_{i-1} \rangle$（事件$E_1$），定义事件$E_2$：第$i$次循环恰好将$x_i$放到了第$i$个位置。则生成上述排列的概率是：

$$
\begin{aligned}
\Pr\{E_2 \cap E_1\} &= \Pr\{E_2 \mid E_1\} \Pr\{E_1\} \\
&= \frac{1}{n-i+1} \cdot \frac{(n-i+1)!}{n!} \\
&= \frac{(n-i)!}{n!} .
\end{aligned}
$$

放在第i个位置的元素是在$[i,n]$中选的

循环不变式

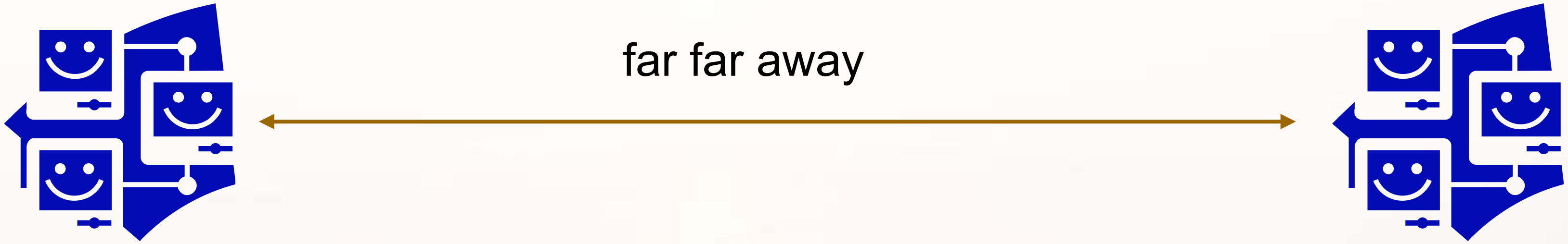
问题8:

招聘问题和生成随机序列问题的算法在分析时关注的问题有什么不一样? 你认为为什么会有这样的差别?

A randomized algorithm is often the simplest and most efficient way to solve a problem. We shall use randomized algorithms occasionally throughout this book.

简单、高效是有代价的，所以随机算法经常用于那些“难”问题，牺牲“一点点”正确性。

一个关于网络通信的例子



We want to verify the consistency between the two databases of size *n* (*n*=10^{16}, say) located on *A* and *B*.

For a deterministic answer, we may have to transfer a message of at least *n* bits, and (!) without an error on the way. It doesn't look a pleasant task.

用随机的方法解决上述问题

- 将逐位比较改为比较两个整数x 和 y的值。
- 并不直接比较它们的值，而是采用以下方法：
  - 计算 $s = x \bmod p$, (p 是一个质数)
  - 计算 $q = y \bmod p$
  - 通过比较 q 和 s 来判断 x 是否等于y 。

- p 是在 $[2, n^2]$ 区间随机选择的质数。

协议描述

Phase 1: R_I chooses a prime p from PRIM(n^2) at random. Every prime from PRIM(n^2) has the same probability $1/\text{Prim}(n^2)$ to be chosen.

Phase 2: R_I computes the integer

$$s = \text{Number}(x) \bmod p$$

(i.e., the remainder of the division $\text{Number}(x) : p$) and sends the binary representations of

$$s \text{ and } p$$

to R_{II}.

Phase 3: After reading s and p, R_{II} computes the number

$$q = \text{Number}(y) \bmod p.$$

If $q \neq s$, then R_{II} outputs "unequal".
If $q = s$, then R_{II} outputs "equal".

问题9：

你能看出这样做的好处吗？

关于通信量

原来最坏情况下要传输 *n* 位信息。

现在只需要传输两个不大于n^2的信息。每个信息的位数为：

$$\lceil \log_2 n^2 \rceil \leq 2 \cdot \lceil \log_2 n \rceil$$

Let us see what that means for $n = 10^{16}$. As already mentioned, the best deterministic protocol cannot avoid the necessity of communicating at least

$$10^{16} \text{ bits}$$

for some inputs. Our protocol WITNESS always works within

$$4 \cdot \lceil \log_2(10^{16}) \rceil \leq 4 \cdot 16 \cdot \lceil \log_2 10 \rceil = 256 \text{ communication bits.}$$

问题10：

你觉得这个结果可靠吗？

假设 $x = 01111_2 = 15$; $y = 10110_2 = 22$; $n=5$

PRIM(25) ={2, 3, 5, 7, 11, 13, 17, 19, 23};

Assume that R_I chooses the prime 5. In Phase 2 computer R_{II} computes

$$s = 15 \bmod 5 = 0$$

and sends the integers $p = 5$ and $s = 0$ to R_{II}. Then R_{II} computes

$$q = 22 \bmod 5 = 2.$$

Since $2 = q \neq s = 0$, the computer R_{II} gives the correct answer "x and y are unequal". ✓

Assume now that R_I chooses the prime 7 from PRIM(25) at random. Then the computer R_I computes

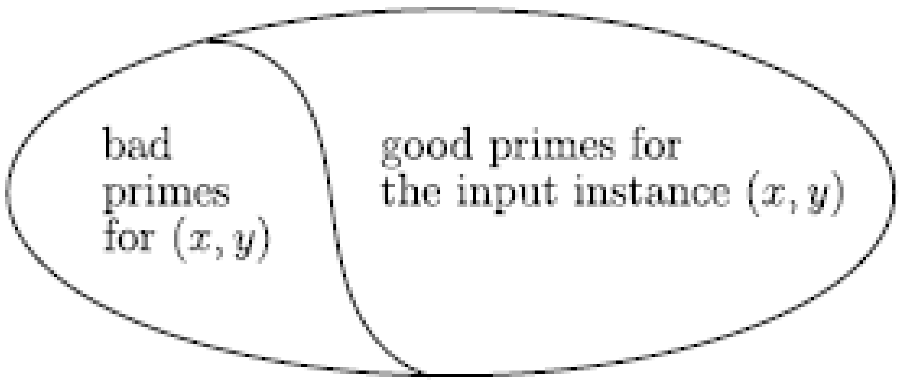
$$s = 15 \bmod 7 = 1$$

and sends the integers $p = 7$ and $s = 1$ to R_{II}. Then R_{II} computes

$$q = 22 \bmod 7 = 1.$$

Since $q = s$, the computer R_{II} gives the wrong answer "x and y are equal". ✗

出错的概率有多大?



$$\text{Error}_{WITNESS}(x,y) = \frac{\text{the number of bad primes for } (x,y)}{\text{Prim}(n^2)}$$

关键是：能否证明这个概率很小?

两个关于质数的结论

对任意*n*>67, $\mathrm{Prim}(m) > \frac{m}{\ln m}$, 因此对任意*n*≥9, $\mathrm{Prim}(n^2) > \frac{n^2}{2\ln n}$

for every problem instance (x,y), the number of bad primes for (x,y) is at most $n-1$,

注意：如果$x=y$, 没有bad prime。

如何识别bad primes?

$$\text{Number}(x) = h_x \cdot p + s, \qquad \text{Number}(y) = h_y \cdot p + s,$$

$$\begin{array}{rr} \text{Number}(x) & h_x \cdot p + s \\ -\text{Number}(y) & -h_y \cdot p - s \\ \hline \text{Dif}(x,y) & h_x \cdot p - h_y \cdot p \end{array}$$

$$\text{Dif}(x,y) = \text{Number}(x) - \text{Number}(y) = h_x \cdot p - h_y \cdot p = (h_x - h_y) \cdot p.$$

你能得出什么有用的结论吗?

A prime p is bad for (x, y) if and only if p divides Dif(x, y)

Bad Primes数量的上限

显然 $\mathrm{Dif}(x, y) = \mathrm{Number}(x) - \mathrm{Number}(y) < 2^n$

你还记得“算术基本定理”吗？

each positive integer larger than 1 can be unambiguously expressed as a product of primes.

$$\begin{aligned}\mathrm{Dif}(x, y) = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} \cdot \ldots \cdot p_k^{i_k} &\geq p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_k \\ &> 1 \cdot 2 \cdot 3 \cdot \ldots \cdot k \\ &= k!\end{aligned}$$

Since $n! > 2^n$ for $n \geq 4$, k must be smaller than n and in this way we have obtained the stated aim that

$$k \leq n - 1,$$

出错率大致的概念

$$\text{Error}_{WITNESS}(x,y) = \frac{\text{the number of bad primes for } (x,y)}{\text{Prim}(n^2)} \leq \frac{n-1}{n^2/\ln n^2} \leq \frac{2\ln n}{n}.$$

Hence, the error probability of WITNESS for problem instances (x, y) with $x \neq y$ is at most $2\ln/n$, which is for $n = 10^{16}$ at most

$$\frac{0.36841}{10^{14}}.$$

问题11：

如果x,y确实相等，则没有“bad prime”。这个结论对降低出错率有什么意义？

问题12：

经证明正确的确定算法
不会出错吗？

课外作业

- CS pp.340-: 4, 8
- CS pp.355-: 1, 2, 4, 6, 12, 16, 18
- TC pp.122-: Ex.5.2-1, 5.2-2, 5.2-4
- TC pp.128-: Ex.5.3-1 – Ex.5.3-4
- TC pp.143-: prob.5-2