计算机问题求解 – 论题1–10 - 函数

2013年12月17日

检查

Exercise 16.6.
We already have an example to show that, with the notation from the theorem above, we need not have $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$. But what is wrong with the following proof of this "non-fact"?

Not a proof.
It follows from Theorem 16.5 that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. To show the reverse set inclusion, we let $y \in f(A_1) \cap f(A_2)$. By definition of intersection, $y \in f(A_1)$ and $y \in f(A_2)$. Therefore, $y = f(x)$ for some x in A_1 and $y = f(x)$ for some $x \in A_2$. Since $x \in A_1$ and $x \in A_2$, we see that $x \in A_1 \cap A_2$. Thus $y = f(x)$ where $x \in A_1 \cap A_2$, so $y \in f(A_1 \cap A_2)$. This proves that $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$, and the non-fact is established! □

问题1:

“函数”与“关系”有什么异同?

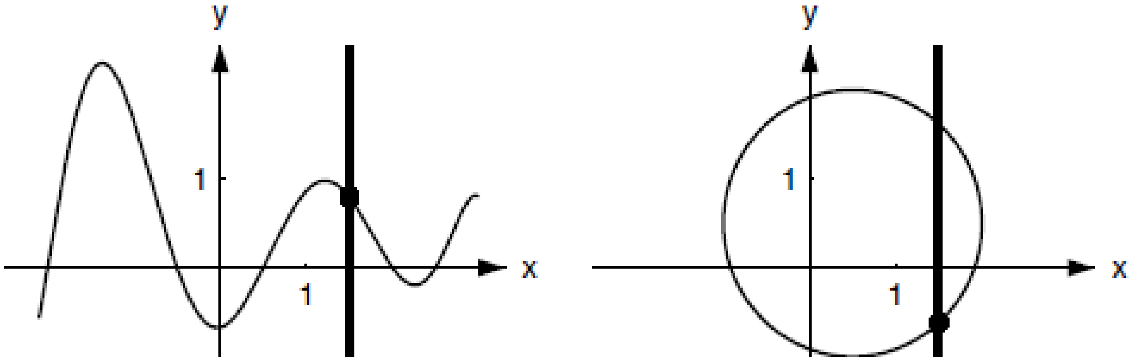
“函数”与“集合”是什么关系?

函数的型构：

$f{:}R\times R\to R\times R$, $f(\langle x,y\rangle) = \langle x+y, x-y\rangle$

问题2：

这里的function与你中学时熟悉的函数有什么异同？



You probably learned that a function $f : \mathbb{R} \to \mathbb{R}$ can be represented by a graph, and that there is a vertical line test to determine whether or not f is a function (See Figure above) . Which condition in the definition corresponds to the vertical line test? Why?

问题3：
你是否能解释一下？

When you define a new mathematical concept, it's always a good idea to think about it and pose questions. Of course, it's also a good idea to answer those questions, if you can. We now turn to some questions that we find interesting. See if you can think of some questions on your own.

问题4:

书中提出了什么问题？你想出了什么“自己”的问题吗？

函数相等到底是什么含义?

函数作为关系，会让你想起什么“问题”？

函数作为集合，会让你想起什么“问题”？

Given functions $f : A \to B$ and $g : C \to D$ with $\operatorname{ran}(f) \subseteq C$, we can define a third function called the **composite function** from A to D. (We will usually call this the **composition**, rather than the composite function.) This composition is the function $g \circ f : A \to D$ defined by $(g \circ f)(x) = g(f(x))$.

Let R be a relation from A to B and S be a relation from B to C. Then we can define a relation, the composition of R and S written as SoR. The relations SoR is a relation from the set A to the set C and is defined as follows:

If $a \in A$, and $c \in A$, then $(a, c) \in S o R$ if and only if for some $b \in B$, we have $(a, b) \in R$ and $(b, c) \in S$.

问题5: 这两个定义有什么关联?

函数的复合运算是否也满足结合律？

$g\circ(f\circ h) = (g\circ f)\circ h$

如何证明这个定律？

问题6：

关于函数自变量的集合只有一个 (domain)，关于函数值的集合却有两个 (codomain和range)，为什么？

问题7：

找到一个函数的range，其实并不容易！

Example 13.7.

Let $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ be defined by $f(x) = (x + 1)/(x - 1)$. Determine the range of f.

Proof.

We will show that $\operatorname{ran}(f) = \mathbb{R} \setminus \{1\}$. Let $y \in \operatorname{ran}(f)$. Then, clearly, $y \in \mathbb{R}$. So $\operatorname{ran}(f) \subseteq \mathbb{R}$. To show that $y \neq 1$, suppose that this is not the case; so we will suppose $y = 1 \in \operatorname{ran}(f)$ and see what happens. Since $y \in \operatorname{ran}(f)$, there exists a point x in the domain with $f(x) = y = 1$. Using the definition of f, we find that $1 = f(x) = (x+1)/(x-1)$. Therefore, $x + 1 = x - 1$. This would mean that $1 = -1$, which is not possible. So $y \in \operatorname{ran}(f)$ implies $y \in \mathbb{R}$ and $y \neq 1$. Thus, $\operatorname{ran}(f) \subseteq \mathbb{R} \setminus \{1\}$.

Now let $y \in \mathbb{R} \setminus \{1\}$. Let $x = (y+1)/(y-1)$. Since $y \neq 1$, we see that $x \in \mathbb{R}$. Remember that we need to check that $x \in \operatorname{dom}(f)$. We know that $x \in \mathbb{R}$. Could we possibly have $x = 1$? Suppose we do, then $1 = (y+1)/(y-1)$ which implies $y - 1 = y + 1$. Thus we would have $-1 = 1$, which is impossible. So $x \in \operatorname{dom}(f)$ and we can evaluate f at x to obtain

$$f(x) = \frac{\frac{y+1}{y-1} + 1}{\frac{y+1}{y-1} - 1} = \frac{y + 1 + y - 1}{y + 1 - y + 1} = y.$$

It follows that $\mathbb{R}\setminus\{1\} \subseteq \operatorname{ran}(f)$. Therefore $\operatorname{ran}(f) = \mathbb{R}\setminus\{1\}$, completing the proof. ■

几种特殊的函数

- **满射onto**
  - f:A→B是满射的：ran$f$=B, iff. ∀y∈B, ∃x∈A, 使得 f(x)=y
- **单射（one to one）**
  - f:A→B是单射的：∀y∈ ran$f$, ∃!x∈A, 使得f(x)=y iff. $\forall x_1,x_2 \in A$, 若$x_1 \neq x_2$，则$f(x_1) \neq f(x_2)$ iff. $\forall x_1,x_2 \in A$, 若 $f(x_1) = f(x_2)$，则$x_1=x_2$。
- **双射（一一对应的）**
  - 满射+单射

几种特殊的函数：例子

- f:$R \to R$, $f(x) = -x^2+2x-1$
- f:$Z^+ \to R$, $f(x) = \ln x$, 单射
- f:$R \to Z$, $f(x) = \lfloor x \rfloor$, 满射
- f:$R \to R$, $f(x) = 2x-1$，双射
- f:$R^+ \to R^+$, $f(x) = (x^2+1)/x$
  - 注意：$f(x) \geq 2$, 而对任意正实数x，$f(x)=f(1/x)$
- f:$R \times R \to R \times R$, $f(<x,y>) = <x+y, x-y>$, 双射。
- f:$N \times N \to N$, $f(<x,y>) = | x^2-y^2 |$

问题8：为什么？

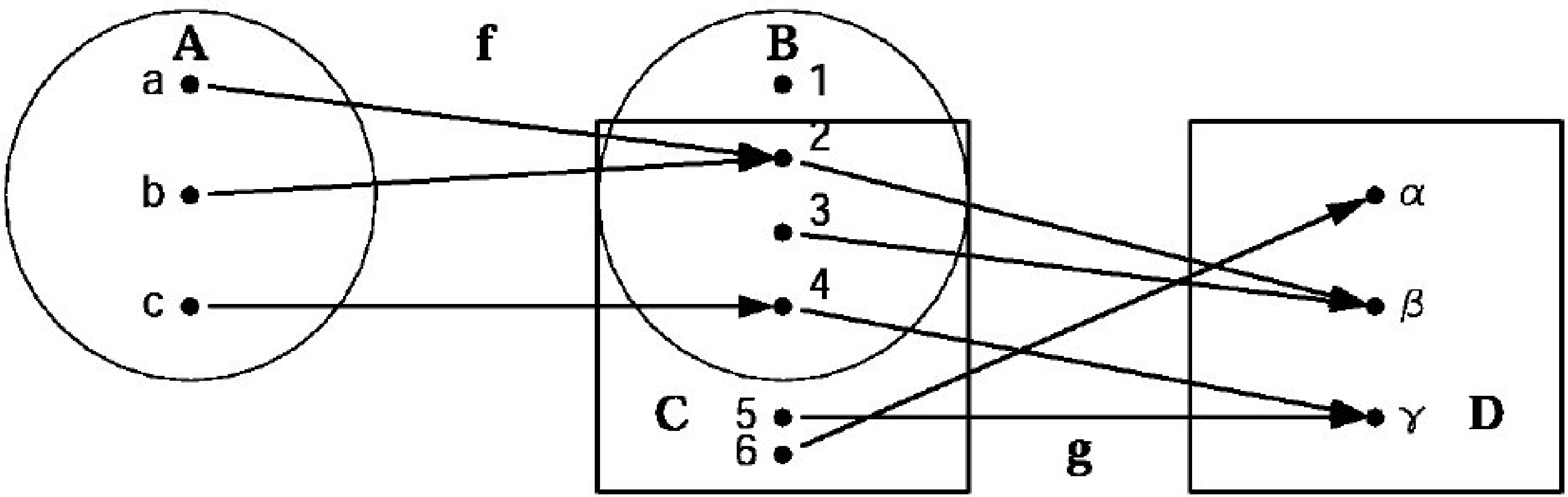
有限集合上一一对应的函数的例子

- S={1,2,3}, 可以在S上定义6个不同的一一对应的函数 (每一个称为一个“置换”):

$$e=\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\quad \alpha=\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}\quad \beta=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$$

$$\gamma=\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}\quad \delta=\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}\quad \varepsilon=\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$$

函数的复合



$$g \circ f : A \rightarrow D$$

α = { (1,2), (2,3), (3,1) };

δ = { (1,3), (2,2), (3,1) };

$\therefore$ $\delta\theta\alpha$ = { (1,2), (2,1), (3,3) } = ε

问题9:

δ□α 和 (δ□α)(X) 有什么不同?

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \delta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \varepsilon = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

问题10:

你能否讨论一下函数复合与函数性质之间的关联?

复合运算*保持*函数性质：单射

- 单射的复合是单射
- 定理：如果$f:A\to B$, $g:B\to C$均是单射，则$g\circ f:A\to C$也是单射。
  - 证明要点：

若不然，即存在$x_1,x_2\in A$, 且$x_1\neq x_2$，使得$g\circ f(x_1)=g\circ f(x_2)$，

设 $f(x_1)=t_1$, $f(x_2)=t_2$,

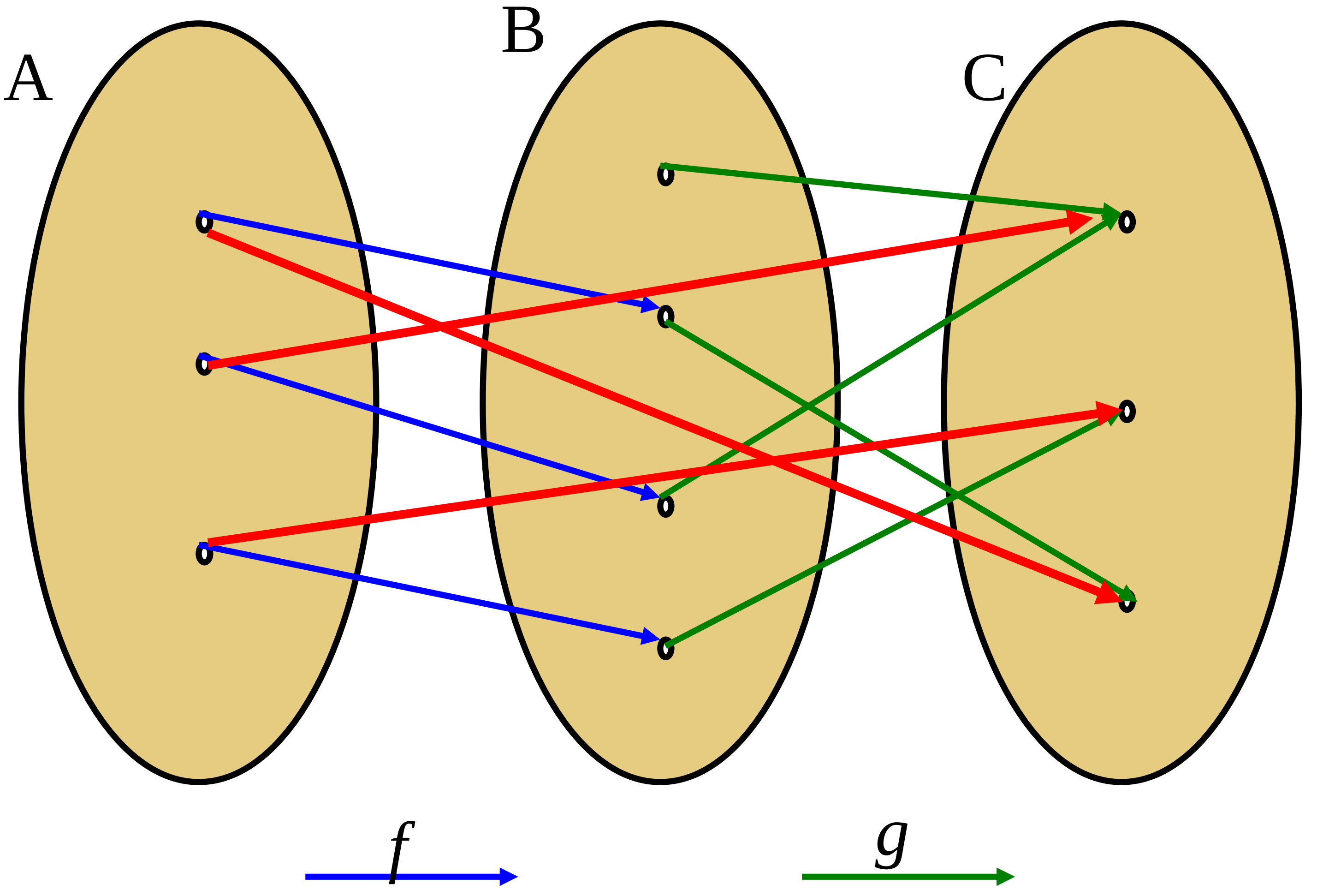
如果 $t_1=t_2$，与f是单射矛盾。

如果 $t_1\neq t_2$，与g是单射矛盾。

但是...

- 若$g \circ f$是单射，能推出f和g是单射吗？
- 显然，f*一定*是单射。

- 若存在$t_1, t_2 \in B$, $t_1 \neq t_2$, 但$g(t_1)=g(t_2)$, (即：**g不是单射！**) 只要 t_1或者t_2不在f值域内，则$g \circ f$ 仍然可能是单射。


A
B
C
f
g

关于反函数

关系的逆　　　函数的反

Undo

问题11:

为什么函数存在反函数的充分必要条件是该函数是bijection?

换一个角度看“undo”。

Example 15.1.
We define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3 - 5$. Graph the function f. Then prove that f is one-to-one and onto. Once you have done that, decide what f^{-1} is.

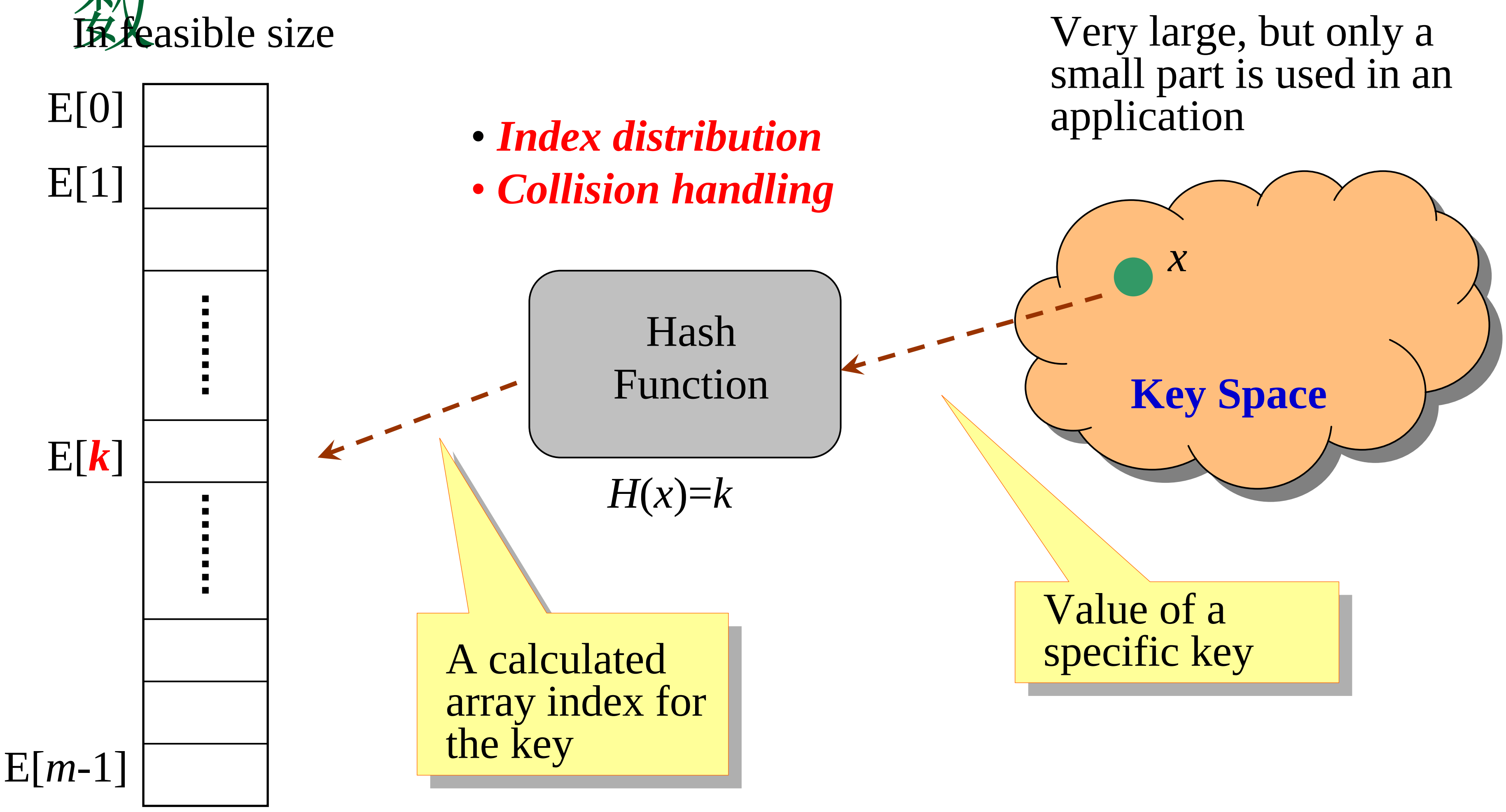
How to show that the reverse of f is: $g(x) = (x + 5)^{1/3}$

Hint:

Let $f : A \to B$ be a bijective function. The **inverse** of f is the function $f^{-1} : B \to A$ defined by

$$f^{-1}(y) = x \text{ if and only if } f(x) = y.$$

Hashing: 计算机科学中的多对一函数



课外作业

- UD 13.3-13.5, 13.11, 13.13;
- UD 14.8, 14.12, 14.13, 14.15;
- UD 15.1, 15.6, 15.7, 15.11-15.15; 15.20
- UD 16.19-16.22

- UD 27.6 (可选)