

- 教材讨论
 - JH第2章第3节第2小节定义2.3.2.1和2.3.2.2及相应的示例

问题1：字母表、词、语言

- alphabet、symbol、word、language
 - 它们是如何被形式化定义的？
 - 你能举出一些实际生活中的例子吗？
- 你能设计一种语言来编码全班同学问求的期末考试成绩吗？
 - 你设计的字母表、词和语言分别是什么？
- 你能设计一种语言来编码图片吗？
 - 你设计的字母表、词和语言分别是什么？
- 编码视频呢？
 - 你设计的字母表、词和语言分别是什么？
- 什么是concatenation of word？
你能利用这个概念来定义这些新概念吗？
 - prefix/suffix/subword
 - concatenation of language

问题1：字母表、词、语言 (续)

Definition 2.3.1.10. Let $\Sigma = \{s_1, s_2, \dots, s_m\}$, $m \geq 1$, be an alphabet, and let $s_1 < s_2 < \dots < s_m$ be a linear ordering on Σ . We define the **canonical ordering** on Σ^* as follows. For all $u, v \in \Sigma^*$,

$$\begin{aligned} u < v \text{ if } &|u| < |v| \\ &\text{or } |u| = |v|, u = xs_iu', \text{ and } v = xs_jv' \\ &\text{for some } x, u', v' \in \Sigma^*, \text{ and } i < j. \end{aligned}$$

- 我们为什么需要一种词的排序规则？
- 你能解释这条排序规则吗？
- 你能给出一种不同的排序规则吗？

问题2：判定和优化问题

Definition 2.3.2.1. A decision problem is a triple (L, U, Σ) where Σ is an alphabet and $L \subseteq U \subseteq \Sigma^*$. An algorithm A solves (decides) the decision problem (L, U, Σ) if, for every $x \in U$,

- (i) $A(x) = 1$ if $x \in L$, and
- (ii) $A(x) = 0$ if $x \in U - L$ ($x \notin L$).

- decision problem中的三个符号分别表示什么意思？
 - 这里的word是什么？
- 判定算法应该给出怎样的结果？

An equivalent form of a description of a decision problem is the following form that specifies the input-output behavior.

Problem (L, U, Σ)

Input: An $x \in U$.
Output: "yes" if $x \in L$,
"no" otherwise.

For many decision problems (L, U, Σ) we assume $U = \Sigma^*$. In that case we shall use the short notation (L, Σ) instead of (L, Σ^*, Σ) .

- 你理解这段话的含义了吗？

问题2：判定和优化问题 (续)

- 这些判定问题分别是什么含义？它们的L分别是什么？

- Primality testing $\{w \in \{0,1\}^* \mid \text{Number}(w) \text{ is a prime}\}$
- Equivalence problem for polynomials
- Satisfiability problem $\{w \in \Sigma_{\text{logic}}^+ \mid w \text{ is a code of a satisfiable formula in CNF}\}$
- Clique problem $\{x\#w \in \{0,1,\#\}^* \mid x \in \{0,1\}^* \text{ and } w \text{ represents a graph that contains a clique of size } \text{Number}(x)\}$
- Vertex cover problem $\{u\#w \in \{0,1,\#\}^+ \mid u \in \{0,1\}^+ \text{ and } w \text{ represents a graph that contains a vertex cover of size } \text{Number}(u)\}$
- Hamiltonian cycle problem $\{w \in \{0,1,\#\}^* \mid w \text{ represents a graph that contains a Hamiltonian cycle}\}$
- Existence of a solution of linear integer programming $\{(A,b) \in \{0,1,\#\}^* \mid \text{Sol}_{\mathbb{Z}}(A,b) \neq \emptyset\}$

问题2：判定和优化问题 (续)

Definition 2.3.2.2. An optimization problem is a 7-tuple $U = (\Sigma_I, \Sigma_O, L, L_I, \mathcal{M}, \text{cost}, \text{goal})$, where

- (i) Σ_I is an alphabet, called the **input alphabet** of U ,
- (ii) Σ_O is an alphabet, called the **output alphabet** of U ,
- (iii) $L \subseteq \Sigma_I^*$ is the **language of feasible problem instances**,
- (iv) $L_I \subseteq L$ is the **language of the (actual) problem instances of U** ,
- (v) \mathcal{M} is a function from L to $\text{Pot}(\Sigma_O^*)$,³⁰ and, for every $x \in L$, $\mathcal{M}(x)$ is called the **set of feasible solutions** for x ,
- (vi) cost is the **cost function** that, for every pair (u, x) , where $u \in \mathcal{M}(x)$ for some $x \in L$, assigns a positive real number $\text{cost}(u, x)$,
- (vii) $\text{goal} \in \{\text{minimum}, \text{maximum}\}$.

- optimization problem中的七个符号分别表示什么意思？

An algorithm A is **consistent** for U if, for every $x \in L_I$, the output $A(x) \in \mathcal{M}(x)$. We say that an algorithm B **solves** the optimization problem U if

- (i) B is consistent for U , and
- (ii) for every $x \in L_I$, $B(x)$ is an optimal solution for x and U .

- 优化算法应该给出怎样的结果？

问题2：判定和优化问题 (续)

- 你能简述minimum vertex cover problem的含义吗？

Input: A graph $G = (V, E)$.

Constraints: $\mathcal{M}(G) = \{S \subseteq V \mid \text{every edge of } E \text{ is incident to at least one vertex of } S\}$.

Cost: For every $S \in \mathcal{M}(G)$, $cost(S, G) = |S|$.

Goal: *minimum*.

- 它和判定问题中的vertex cover problem之间有什么联系？

$\{u\#w \in \{0, 1, \#\}^+ \mid u \in \{0, 1\}^+ \text{ and } w \text{ represents a graph that contains a vertex cover of size } Number(u)\}$

问题2：判定和优化问题 (续)

- 你能简述maximum clique problem的含义吗？

Input: A graph $G = (V, E)$

Constraints: $\mathcal{M}(G) = \{S \subseteq V \mid \{\{u, v\} \mid u, v \in S, u \neq v\} \subseteq E\}$.

$\{\mathcal{M}(G) \text{ contains all complete subgraphs (cliques) of } G\}$

Costs: For every $S \in \mathcal{M}(G)$, $cost(S, G) = |S|$.

Goal: *maximum*.

- 它和判定问题中的clique problem之间有什么联系？

$\{x\#w \in \{0, 1, \#\}^* \mid x \in \{0, 1\}^* \text{ and } w \text{ represents a graph}$

that contains a clique of size $Number(x)\}$

问题2：判定和优化问题 (续)

- 你能简述maximum cut problem的含义吗？

Input: A graph $G = (V, E)$.

Constraints:

$$\mathcal{M}(G) = \{(V_1, V_2) \mid V_1 \cup V_2 = V, V_1 \neq \emptyset \neq V_2, \text{ and } V_1 \cap V_2 = \emptyset\}.$$

Costs: For every cut $(V_1, V_2) \in \mathcal{M}(G)$,

$$\text{cost}((V_1, V_2), G) = |E \cap \{\{u, v\} \mid u \in V_1, v \in V_2\}|.$$

Goal: *maximum*.

- 你能给出一个与之相关的判定问题吗？

问题2：判定和优化问题 (续)

- 你能简述traveling salesperson problem的含义吗？

Input: A weighted complete graph (G, c) , where $G = (V, E)$ and $c : E \rightarrow \mathbb{N}$. Let $V = \{v_1, \dots, v_n\}$ for some $n \in \mathbb{N} - \{0\}$.

Constraints: For every input instance (G, c) , $\mathcal{M}(G, c) = \{v_{i_1}, v_{i_2}, \dots, v_{i_n}, v_{i_1} \mid (i_1, i_2, \dots, i_n) \text{ is a permutation of } (1, 2, \dots, n)\}$, i.e., the set of all Hamiltonian cycles of G .

Costs: For every Hamiltonian cycle $H = v_{i_1} v_{i_2} \dots v_{i_n} v_{i_1} \in \mathcal{M}(G, c)$,
 $cost((v_{i_1}, v_{i_2}, \dots, v_{i_n}, v_{i_1}), (G, c)) = \sum_{j=1}^n c(\{v_{i_j}, v_{i_{(j \bmod n)+1}}\})$,
i.e., the cost of every Hamiltonian cycle H is the sum of the weights of all edges of H .

Goal: *minimum*.

问题2：判定和优化问题 (续)

- 你能简述knapsack problem的含义吗？

Input: A positive integer b , and $2n$ positive integers $w_1, w_2, \dots, w_n, c_1, c_2, \dots, c_n$ for some $n \in \mathbb{N} - \{0\}$.

Constraints:

$$\mathcal{M}(b, w_1, \dots, w_n, c_1, \dots, c_n) = \{T \subseteq \{1, \dots, n\} \mid \sum_{i \in T} w_i \leq b\}.$$

Costs: For each $T \in \mathcal{M}(b, w_1, \dots, w_n, c_1, \dots, c_n)$,

$$\text{cost}(T, b, w_1, \dots, w_n, c_1, \dots, c_n) = \sum_{i \in T} c_i.$$

Goal: *maximum*.

问题2：判定和优化问题 (续)

- 你能简述bin-packing problem的含义吗？

Input: n rational numbers $w_1, w_2, \dots, w_n \in [0, 1]$ for some positive integer n .

Constraints: $\mathcal{M}(w_1, w_2, \dots, w_n) = \{S \subseteq \{0, 1\}^n \mid \text{for every } s \in S, s^T \cdot (w_1, w_2, \dots, w_n) \leq 1, \text{ and } \sum_{s \in S} s = (1, 1, \dots, 1)\}$.

{If $S = \{s_1, s_2, \dots, s_m\}$, then $s_i = (s_{i1}, s_{i2}, \dots, s_{in})$ determines the set of objects packed in the i th bin. The j th object is packed into the i th bin if and only if $s_{ij} = 1$. The constraint

$$s_i^T \cdot (w_1, \dots, w_n) \leq 1$$

assures that the i th bin is not overfilled. The constraint

$$\sum_{s \in S} s = (1, 1, \dots, 1)$$

assures that every object is packed in exactly one bin.}

Cost: For every $S \in \mathcal{M}(w_1, w_2, \dots, w_n)$,

$$\text{cost}(S, (w_1, \dots, w_n)) = |S|.$$

Goal: *minimum*.

问题2：判定和优化问题 (续)

- 你能简述makespan scheduling problem的含义吗？

Input: Positive integers p_1, p_2, \dots, p_n and an integer $m \geq 2$ for some $n \in \mathbb{N} - \{0\}$.

$\{p_i$ is the processing time of the i th job on any of the m available machines}.

Constraints: For every input instance (p_1, \dots, p_n, m) of MS,

$\mathcal{M}(p_1, \dots, p_n, m) = \{S_1, S_2, \dots, S_m \mid S_i \subseteq \{1, 2, \dots, n\}$ for $i = 1, \dots, m$, $\bigcup_{k=1}^m S_k = \{1, 2, \dots, n\}$, and $S_i \cap S_j = \emptyset$ for $i \neq j\}$.

$\{\mathcal{M}(p_1, \dots, p_n, m)$ contains all partitions of $\{1, 2, \dots, n\}$ into m subsets. The meaning of (S_1, S_2, \dots, S_m) is that, for $i = 1, \dots, m$, the jobs with indices from S_i have to be processed on the i th machine}.

Costs: For each $(S_1, S_2, \dots, S_m) \in \mathcal{M}(p_1, \dots, p_n, m)$,

$cost((S_1, \dots, S_m), (p_1, \dots, p_n, m)) = \max \{\sum_{l \in S_i} p_l \mid i = 1, \dots, m\}$.

Goal: *minimum*.

问题2：判定和优化问题 (续)

- 你能简述set cover problem的含义吗？

Input: (X, \mathcal{F}) , where X is a finite set and $\mathcal{F} \subseteq \text{Pot}(X)$ such that $X = \bigcup_{S \in \mathcal{F}} S$.

Constraints: For every input (X, \mathcal{F}) ,
 $\mathcal{M}(X, \mathcal{F}) = \{C \subseteq \mathcal{F} \mid X = \bigcup_{S \in C} S\}$.

Costs: For every $C \in \mathcal{M}(X, \mathcal{F})$, $\text{cost}(C, (X, \mathcal{F})) = |C|$.

Goal: *minimum*.

问题2：判定和优化问题 (续)

- 你能简述maximum satisfiability problem的含义吗？

Input: A formula $\Phi = F_1 \wedge F_2 \wedge \dots \wedge F_m$ over $X = \{x_1, x_2, \dots\}$ in CNF
(an equivalent description of this instance of MAX-SAT is to consider the set of clauses F_1, F_2, \dots, F_m).

Constraints: For every formula Φ over the set $\{x_1, \dots, x_n\} \subseteq X, n \in \mathbb{N} - \{0\}$,
 $\mathcal{M}(\Phi) = \{0, 1\}^n$.
{Every assignment of values to $\{x_1, \dots, x_n\}$ is a feasible solution,
i.e., $\mathcal{M}(\Phi)$ can also be written as $\{\alpha \mid \alpha : X \rightarrow \{0, 1\}\}$.

Costs: For every Φ in CNF, and every $\alpha \in \mathcal{M}(\Phi)$,
 $cost(\alpha, \Phi)$ is the number of clauses satisfied by α .

Goal: *maximum*.

问题2：判定和优化问题 (续)

- 你能简述integer linear programming的含义吗？

Input: An $m \times n$ matrix $A = [a_{ij}]_{i=1, \dots, m, j=1, \dots, n}$, and two vectors $b = (b_1, \dots, b_m)^\top$, $c = (c_1, \dots, c_n)^\top$ for some $n, m \in \mathbb{N} - \{0\}$, a_{ij}, b_i, c_j are integers for $i = 1, \dots, m$, $j = 1, \dots, n$.

Constraints: $\mathcal{M}(A, b, c) = \{X = (x_1, \dots, x_n) \in \mathbb{Z}^n \mid AX = b \text{ and } x_i \geq 0 \text{ for } i = 1, \dots, n\}$.

Costs: For every $X = (x_1, \dots, x_n) \in \mathcal{M}(A, b, c)$,
 $cost(X, (A, b, c)) = \sum_{i=1}^n c_i x_i$.

Goal: *minimum*.

问题2：判定和优化问题 (续)

- 你能简述maximum linear equation problem mod k 的含义吗？

Input: A set S of m linear equations over n unknowns, $n, m \in \mathbb{N} - \{0\}$,
with coefficients from \mathbb{Z}_k .

(An alternative description of an input is an $m \times n$ matrix over \mathbb{Z}_k
and a vector $b \in \mathbb{Z}_k^m$).

Constraints: $\mathcal{M}(S) = \mathbb{Z}_k^m$
{a feasible solution is any assignment of values from $\{0, 1, \dots, k-1\}$
to the n unknowns (variables)}.

Costs: For every $X \in \mathcal{M}(S)$,
 $cost(X, S)$ is the number of linear equations of S satisfied by X .

Goal: *maximum*.