计算机问题求解 – 论题2-2
- 组合与计数

Part I
算法分析与计数

问题1：
通常在讨论算法时间代价时我们“数”什么？

“critical operation”:
假设：总代价与critical operations的代价成正比
选择的“合理性”

Selection sort，本质上与“冒泡”一样

顺便问一句: 你能用循环不变量证明此算法正确吗?

```
(1)  for i = 1 to n − 1
(2)      for j = i + 1 to n
(3)          if (A[i] > A[j])
(4)              exchange A[i] and A[j]
```

How many times is the comparison $A[i] > A[j]$ made in Line 3?

critical operation

Principle 1.1 (Sum Principle)

The size of a union of a family of mutually disjoint finite sets is the sum of the sizes of the sets.

$$\left|\bigcup_{i=1}^{m} S_i\right| = \sum_{i=1}^{m} |S_i|.$$

问题2：你如何理解这里所体现的"抽象"过程？

我们究竟是在数什么？

If our goal were to solve only Exercise 1.1-1, then our abstraction would have been almost a mindless exercise that complicated what was an "obvious" solution. However, the sum principle will prove to be useful in a variety of problems. Thus, **the value of abstraction is that recognizing the abstract elements of a problem often helps us solve subsequent problems.**

你能解释一下抽象的过程吗?

矩阵相乘

```
(1)  for i = 1 to r
(2)      for j = 1 to m
(3)          S = 0
(4)          for k = 1 to n
(5)              S = S + A[i,k] * B[k,j]
(6)          C[i,j] = S
```

问题3:

多少集合,
什么关系?

How many multiplications (expressed in terms of r, m, and n) does this pseudocode carry out in total among all the iterations of Line 5?

$$T_i = \bigcup_{j=1}^{m} S_j.$$

$$|T_i| = \left|\bigcup_{j=1}^{m} S_j\right| = \sum_{j=1}^{m} |S_j| = \sum_{j=1}^{m} n = mn.$$

Principle 1.3 (Product Principle)

The size of a union of m disjoint sets, each of size n, is mn.

分块计数

The first loop makes n(n + 1)/2−1 Comparisons.

Ask yourself first where the n(n + 1)/2 comes from and then why we subtracted 1

❶

```
 (1)  for i = 1 to n − 1
 (2)       minval = A[i]
 (3)       minindex = i
 (4)       for j = i to n
 (5)            if (A[j] < minval)
 (6)                 minval = A[j]
 (7)                 minindex = j
 (8)       exchange A[i] and A[minindex]
```

❷

```
 (9)  bigjump = 0
(10)  for i = 2 to n
(11)       if (A[i] > 2 * A[i − 1])
(12)            bigjump = bigjump + 1
```

❶就是排序

❷数“大间隔”

How many comparisons does the pseudocode make in Lines 5 and 11?

操作计数与子集计数

相同的情况，不同的抽象：

在排序的例子里，对任意含两个元素的子集，我们做一次比较，则比较次数等于n个元素的集合所有的两个元素的子集的个数。

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

注意: 数操作实际上是数“有序对”，而子集内元素是无序的。

多少种密码?

A password for a certain computer system is supposed to be between four and eight characters long and composed of lowercase and/or uppercase letters. How many passwords are possible? What counting principles did you use? Estimate the percentage of the possible passwords that have exactly four letters.

问题4:

几个集合？相加？相乘？

问题5:
每个集合大小怎么算?

$$52^4+52^5+52^6+52^7+52^8.$$

你能将这个计算推广到一般的原理吗?

从数list到数函数

Principle 1.4 (Product Principle, Version 2)

If a set S of lists of length m has the properties that

1. there are i_1 different first elements of lists in S, and
2. for each $j > 1$ and each choice of the first $j-1$ elements of a list in S, there are i_j choices of elements in position j of those lists,

then there are $i_1 i_2 \cdots i_m = \prod_{k=1}^{m} i_k$ lists in S.

问题6：通俗地说说这是什么意思？

假设list是一步一步生成的，每“步”有多少种选择，全部乘起来。

问题7：这与数函数有什么关系？

平面上 n 个点能生成多少三角形

```
(1)  trianglecount = 0
(2)  for i = 1 to n
(3)      for j = i+1 to n
(4)          for k = j+1 to n
(5)              if points i, j, and k are not collinear
(6)                  trianglecount = trianglecount + 1
```

Among all iterations of line 5 of the pseudocode, what is the total number of times this line checks three points to see if they are collinear?

前面排序算法可以通过数子集个数来计数，有什么启发吗？

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$$

问题8:

你能解释这个原理的应用吗?

```
(1)  trianglecount = 0
(2)  for i = 1 to n
(3)        for j = i+1 to n
(4)              for k = j+1 to n
(5)                    if points i, j, and k are not collinear
(6)                          trianglecount = trianglecount + 1
```

注意这里的三个循环变量。

Among all iterations of line 5 of the pseudocode, what is the total number of times this line checks three points to see if they are collinear?

Principle 1.5 (Bijection Principle)

Two sets have the same size if and only if there is a one-to-one function from one set onto the other.

问题9：

你能否简单叙述一下：为什么在n个数中任取3个不同的数构成的递增序列的集合与所有3个数字构成的子集的集合是等势的？

双射：$(i, j, k) \to \{i, j, k\}$

What bijection is behind our assertion that the number of increasing triples equals the number of three-element subsets? We define the function f as the function that takes the increasing triple (i, j, k) to the subset $\{i, j, k\}$. Because the three elements of an increasing triple are different, the subset is a three-element set; so, we have a function from increasing triples to three-element sets. Because two different triples can't be the same set in two different orders, they must be associated with different sets. Thus, f is one-to-one. Because each set of three integers can be listed in increasing order, it is thus the image of an increasing triple under f. Therefore f is onto. Thus, we have a one-to-one correspondence, or bijection, between the set of increasing triples and the set of three-element sets.

三步:

- 建立一个函数；（注意：不是任意关系都是函数）
- 证明：这个函数是一对一的
- 证明：这个函数是满射

Part II

利用等价关系计数

问题10：

你还记得什么是等价关系吗？它和集合分划有什么关系？

给定集合A上的等价关系，所有的等价类构成集合A的一个分划。

如何将等价关系与计数相关联

问题：

用2种颜色给5个对象着色，并保证每种颜色最少用于2个对象，有多少种不同的着色法？

问题11：

这个问题其实非常简单，你能直接给出答案吗？

着色总数：$\binom{5}{3}=\binom{5}{2}=10$

但是如果我们定义一种等价关系，使得等价类的个数就是不同的着色方案的个数，那就可以通过“抽象”推出一个更加有用的解法。

5个对象任意排列方式有5！=120种，如果用颜色代码加标号，有哪些可以被认为是“相同”的着色方案呢？

给出一个等价类

$\{A, B, C, D, E\}$. Consider the particular labeling in which A, B, and D are labeled blue and C and E are labeled red. Which lists correspond to this labeling? They are

$$\begin{array}{cccccc} ABDCE & ABDEC & ADBCE & ADBEC & BADCE & BADEC \\ BDACE & BDAEC & DABCE & DABEC & DBACE & DBAEC, \end{array}$$

(Quotient Principle)

If an equivalence relation on a p-element set S has q classes each of size r, then $q = p/r$.

这里： p=120, r=12, q=120/12=10

你能解释下列问题的等价类吗

When four people sit down at a round table to play cards, two lists of their four names are equivalent as seating charts if each person has the same person to the right in both lists.[9] (The person to the right of the person in Position 4 of the list is the person in Position 1.) We use Theorem 1.7 to count the number of possible ways to seat the players. We take our set S to be the set of all four-element permutations of the four people, that is, the set of all lists of the four people.

We wish to count the number of ways to attach $n > 2$ distinct beads to the corners of a regular n-gon (or string them on a necklace). We say that two lists of the n beads are equivalent if each bead is adjacent to exactly the same beads in both lists. (The first bead in the list is considered to be adjacent to the last.)

$$\frac{n!}{2n} = \frac{(n-1)!}{2}$$

问题12：

为什么上面问题中等价类大小为n(这里是4)，
而下面问题中则为2n?

Multiset：允许相同元素的集合

把k册相同的书放置在n层书架上（每层的空间可以放得下所有的书），有多少种不同的放置方式？为什么可以用multiset建模？

How many k-element multisets can we choose from an n-element set?

问题13:

为什么不能直接采用上述基于等价关系的方法求解？

间接求解：借助书架问题

Suppose, however, that we could count the number of ways to arrange k distinct books on the n shelves of a bookcase. We can still think of the multiplicity of a shelf as being the number of books on it. However, many different arrangements of distinct books will give us the same multiplicity function. In fact, any way of mixing the books among themselves that does not change the number of books on each shelf will give us the same multiplicities. But the number of ways to mix the books among themselves is the number of permutations of the books—namely, $k!$. Thus, it looks like we have an equivalence relation on the arrangements of distinct books on a bookshelf such that

构造出需要的等价类

1. each equivalence class has $k!$ elements, and
2. there is a bijection between the equivalence classes and k-element multisets of the n shelves.

Therefore, if we can compute the number of ways to arrange k *distinct* books on the n shelves of a bookcase, we should be able to apply the quotient principle to compute the number of k-element multisets of an n-element set.

书架上的排列问题

暂且假设每本不同

We have k books to arrange on the n shelves of a bookcase. The order in which the books appear on a shelf matters, and each shelf can hold all the books. We will assume that as the books are placed on the shelves, they are pushed as far to the left as they will go. Thus, all that matters is the order in which the books appear. When book i is placed on a shelf, it can go between two books already there or to the left or right of all the books on that shelf.

$$\underline{n(n+1)(n+2)\cdots(n+k-1)} = \prod_{i=1}^{k}(n+i-1)$$

$$n^{\overline{k}}$$

$$= \prod_{j=0}^{k-1}(n+j) = \frac{(n+k-1)!}{(n-1)!}.$$

随着已放置书增加，可选的位置也在“均匀”增加

回到Multiset问题

对应到书架问题：我们只关心书在各层书架上的分布，可以当作*k*本书是一样的，所以任何一种排列等完全等价。

The number of k-element multisets chosen from an n-element set is

$$\frac{n^{\overline{k}}}{k!} = \binom{n+k-1}{k}.$$