4-2：置换群与拉格朗日定理

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TJ 5-3(d)

Express the following permutations as products of transpositions and identify them as even or odd.

$$(17254)(1423)(154632)$$

$$(14)(15)(12)(17)(13)(12)(14)(12)(13)(16)(14)(15)$$

TJ 5-5: List all subgroups of S_4

$$\boxed{|S_4| = 24 = 2^3 \cdot 3}$$

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- $|X| = 1$:

$$\text{identity: } \{()\}$$

- $|X| = 2$:

$$\langle(1,2)\rangle\,;\langle(1,3)\rangle\,;\langle(1,4)\rangle\,;\langle(2,3)\rangle\,;\langle(2,4)\rangle\,;\langle(3,4)\rangle$$

$$\langle(1,2)(3,4)\rangle\,;\langle(1,3)(2,4)\rangle\,;\langle(1,4)(2,3)\rangle$$

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$$\langle(1,2)(3,4)\rangle\,;\langle(1,3)(2,4)\rangle\,;\langle(1,4)(2,3)\rangle$$

- $|X| = 3$

$$\langle(1,2,3)\rangle\,;\langle(1,2,4)\rangle\,;\langle(1,3,4)\rangle\,;\langle(2,3,4)\rangle$$

TJ 5-5: List all subgroups of S_4

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- $|X| = 4$

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- $|X| = 4$

$$\begin{aligned} Z_4 \quad &\cong \langle(1,2,3,4)\rangle = \langle(1,4,3,2)\rangle = \{(), (1234), (13)(24), (1432)\} \\ &\cong \langle(1,3,2,4)\rangle = \langle(1,4,2,3)\rangle = \{(), (1324), (12)(34), (1423)\} \\ &\cong \langle(1,3,4,2)\rangle = \langle(1,2,4,3)\rangle = \{(), (1342), (14)(23), (1243)\}; \end{aligned}$$

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$$\begin{aligned} \text{Klein four-group } V_4 &\cong \langle (1,2), (3,4) \rangle = \{(), (12)(34), (12), (34)\} \\ &\cong \langle (1,3), (2,4) \rangle = \{(), (13)(24), (13), (24)\} \\ &\cong \langle (1,4), (2,3) \rangle = \{(), (14)(23), (14), (23)\} \end{aligned}$$

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TJ 5-5: List all subgroups of S_4

$$\boxed{|S_4| = 24 = 2^3 \cdot 3}$$

- $|X| = 6$

TJ 5-5: List all subgroups of S_4

$$\boxed{|S_4| = 24 = 2^3 \cdot 3}$$

- $|X| = 6$

$$\begin{aligned}
S_3 &\cong \langle (1,2,3),(1,2) \rangle = \{(),(12),(13),(23),(123),(132)\} \\
&\cong \langle (1,2,4),(1,2) \rangle = \{(),(12),(14),(24),(124),(142)\} \\
&\cong \langle (1,3,4),(1,3) \rangle = \{(),(13),(14),(34),(134),(143)\} \\
&\cong \langle (2,3,4),(2,3) \rangle = \{(),(23),(24),(34),(234),(243)\}
\end{aligned}$$

TJ 5-5: List all subgroups of S_4

$$\boxed{|S_4| = 24 = 2^3 \cdot 3}$$

- $|X| = 8$

TJ 5-5: List all subgroups of S_4

$$\boxed{|S_4| = 24 = 2^3 \cdot 3}$$

- $|X| = 8$

$$\begin{aligned} D_8 &\cong \langle (1,2,3,4),(1,3)\rangle = \langle (1,4,3,2),(1,2)\rangle \\ &= \{(),(1234),(13)(24),(14)(23),(1432),(12)(34),(14)(23),(13),(24)\} \\ &\cong \langle (1,3,2,4),(1,2)\rangle = \langle (1,4,2,3),(1,2)\rangle \\ &\cong \langle (1,3,4,2),(1,4)\rangle = \langle (1,2,4,3),(1,4)\rangle \end{aligned}$$

TJ 5-5: List all subgroups of S_4

$$\boxed{|S_4| = 24 = 2^3 \cdot 3}$$

- $|X| = 12$

https://groupprops.subwiki.org/wiki/Subgroup_structure_of_symmetric_group:S4

TJ 5-5: List all subgroups of S_4

$$\boxed{|S_4| = 24 = 2^3 \cdot 3}$$

- $|X| = 12$

$$A_4 \quad \cong \langle (1,2,3), (1,2)(3,4) \rangle$$

https://groupprops.subwiki.org/wiki/Subgroup_structure_of_symmetric_group:S4

TJ 5-5: List all subgroups of S_4

$$\boxed{|S_4| = 24 = 2^3 \cdot 3}$$

- $|X| = 12$

$$A_4 \quad \cong \langle (1,2,3), (1,2)(3,4) \rangle$$

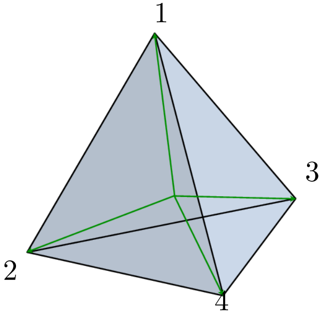
- $|X| = 24$

$$S_4$$

https://groupprops.subwiki.org/wiki/Subgroup_structure_of_symmetric_group:S4

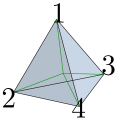
TJ 5-16

Find the group of rigid motions of a **tetrahedron**. Show that this is the same group as A_4.

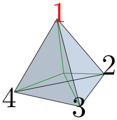


TJ 5-16

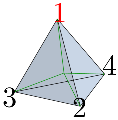
(1)



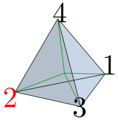
(243)



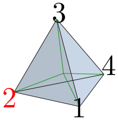
(234)



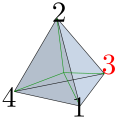
(143)



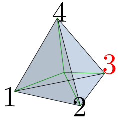
(134)



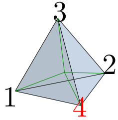
(124)



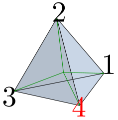
(142)



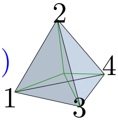
(132)



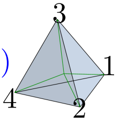
(123)



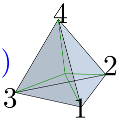
(12)(34)



(13)(24)



(14)(23)



TJ 5-26(b)

Prove that any element in S_n can be written as a finite product of the following permutations.

$$(12), (23), ..., (n-1, n)$$

Any **permutation** of a finite set containing at least two elements can be written as the product of transpositions.

Question:

Given a transposition $(a_x a_y)$, it can be written as a finite product of the following permutations.

$$(12), (23), ..., (n-1, n)$$

证明.

Assume, $x < y$, introduction on $y - x$

证明.

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- **B**: $y - x = 1$, obviously.

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Assume, $x < y$, introduction on $y - x$

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- **H**: Assume it is true for all $y - x < k$

证明.

Assume, $x < y$, introduction on $y - x$

- **B**: $y - x = 1$, obviously.
- **H**: Assume it is true for all $y - x < k$
- **I**: $y - x = k$

$$(a_x a_y) = (a_x a_{x+1})(a_{x+1} a_y)(a_x a_{x+1})$$

By **H**, $(a_{x+1} a_y)$ can be written as a finite product of the required permutations. So $(a_x a_y)$ can be written as a finite product of the required permutations as well.

□

TJ 5-36

Let r and s be the elements in D_n

(a) Show that $srs = r^{-1}$

(b) Show that $r^k s = sr^{-k}$ in D_n

(c) Prove that the order of $r^k \in D_n$ is $n/\gcd(k, n)$

(a) Show that $srs = r^{-1}$

证明.

$$r = \begin{bmatrix} \cos(\pi/n) & -sin(\pi/n) \\ sin(\pi/n) & cos(\pi/n) \end{bmatrix}, s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then,

$$srs = \begin{bmatrix} \cos(\pi/n) & sin(\pi/n) \\ -sin(\pi/n) & cos(\pi/n) \end{bmatrix} = r^{-1}$$

□

(b) Show that $r^k s = sr^{-k}$ in D_n

证明.

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证明.

Introduction on k

(b) Show that $r^k s = sr^{-k}$ in D_n

证明.

Introduction on k

- **B:** $k = 1$, $rs = sr^{-1}$, from (a);

(b) Show that $r^k s = sr^{-k}$ in D_n

证明.

Introduction on k

- **B:** $k = 1$, $rs = sr^{-1}$, from (a);
- **H:** assume it holds for all $k < j$

(b) Show that $r^k s = sr^{-k}$ in D_n

证明.

Introduction on k

- **B:** $k = 1$, $rs = sr^{-1}$, from (a);
- **H:** assume it holds for all $k < j$
- **I:** $k = j$

$$r^j s = r^{j-1} rs = r^{j-1} sr^{-1} = sr^{-(j-1)} r^{-1} = sr^{-j}$$

$\square$

(c) Prove that the order of $r^k \in D_n$ is $n/\gcd(k,n)$

证明.

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证明.

- $\langle r \rangle$ is a cyclic group of order n

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证明.

- $\langle r \rangle$ is a cyclic group of order n
- By Theorem 4.13, we have the proof.

$\square$

Theorem 4.13. *Let G be a cyclic group of order n and suppose that $a \in G$ is a generator of the group. If $b = a^k$, then the order of b is n/d, where $d = \gcd(k, n)$.*

TJ 5-29

Recall that the center of a group G is

$$Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}.$$

Find the center of D_8. What about the center of D_{10}? What is the center of D_n?

$Z(D_8)$ and $Z(D_{10})$

$$Z(D_8) = \{1, r^4\}$$

$$Z(D_{10}) = \{1, r^5\}$$

$Z(D_n)$

$$gx = xg \Leftrightarrow gxg^{-1} = x$$

$Z(D_n)$

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- **Case 1**: $g = r^k$

$Z(D_n)$

$$gx = xg \Leftrightarrow gxg^{-1} = x$$

- **Case 1**: $g = r^k$
  - $a = r^j$

obviously $r^k a r^{-k} = a$

$Z(D_n)$

$$gx = xg \Leftrightarrow gxg^{-1} = x$$

- **Case 1**: $g = r^k$
  - $a = r^j$

$$\text{obviously } r^k a r^{-k} = a$$

  - $a = sr^j$

$$\begin{aligned} r^k a r^{-k} &= r^k s r^j r^{-k} \\ &= s r^{-k} r^j r^{-k} \text{ by TJ 5-36(b)} \\ &= s r^j r^{-2k} \end{aligned}$$

$sr^j = sr^{j-2k}$ iff $r^{-2k} = e$ iff $-2k = 0(\mod n)$
So, if n is even, $k = 0, n/2$; if n is odd, $k = 0$;

$Z(D_n)$

$$gx = xg \Leftrightarrow gxg^{-1} = x$$

- **Case 2**: $g = sr^k$, $0 \leq k < n$

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$$gx = xg \Leftrightarrow gxg^{-1} = x$$

- **Case 2**: $g = sr^k$, $0 \leq k < n$
  - $a = r^j$, $0 \leq j < n$

$$\begin{aligned} sr^k a(sr^k)^{-1} &= sr^k r^j r^{-k} s^{-1} = sr^j s \\ &= ssr^{-j} (\text{ by TJ 5-36(b)}) \\ &= r^{-j} \end{aligned}$$

$r^j = r^{-j}$ iff $r^{2j} = e$ iff $2j = 0 (\mod n)$. Only hold for $n < 2$.

$Z(D_n)$

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- **Case 2**: $g = sr^k$, $0 \le k < n$
  - $a = r^j$, $0 \le j < n$

$$\begin{aligned} sr^k a(sr^k)^{-1} &= sr^k r^j r^{-k} s^{-1} = sr^j s \\ &= ssr^{-j} (\text{ by TJ 5-36(b)}) \\ &= r^{-j} \end{aligned}$$

$r^j = r^{-j}$ iff $r^{2j} = e$ iff $2j = 0 (\mod n)$. Only hold for $n < 2$.
  - $a = sr^j$

$$\begin{aligned} sr^k a(sr^k)^{-1} &= sr^k sr^j r^{-k} s^{-1} = sr^k sr^j r^{-k} s \\ &= sr^k srj - ks \\ &= sr^k ssr^{k-j} (\text{ by TJ 5-36(b)}) \\ &= sr^{2k-j} \end{aligned}$$

$sr^j = sr^{2k-j}$ iff $k = j (\mod n)$. Only hold for $n < 2$

TJ 6-11

Let H be a subgroup of a group G and suppose that $g_1, g_2 \in G$. Prove that the following conditions are equivalent.

(a) $g_1 H = g_2 H$

(b) $H g_1^{-1} = H g_2^{-1}$

(c) $g_1 H \subseteq g_2 H$

(d) $g_2 \in g_1 H$

(e) $g_1^{-1} g_2 \in H$

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$$
\begin{tikzcd}
\text{(a)} \arrow[dd] & & \text{(c)} \arrow[ll] \\
& \text{(b)} & \\
\text{(e)} \arrow[ur, leftrightarrow] \arrow[rr] & & \text{(d)} \arrow[uu]
\end{tikzcd}
$$

(a) → (e)

(a) $g_1H = g_2H$

(b) $Hg_1^{-1} = Hg_2^{-1}$

(c) $g_1H \subseteq g_2H$

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\begin{tikzcd}
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& (b) & \\
(e) \arrow[ur, leftrightarrow] \arrow[rr] & & (d) \arrow[uu]
\end{tikzcd}
$$

证明.

$$
\begin{aligned}
g_1H = g_2H \quad & \Rightarrow g_1^{-1}g_2H = H \\
& \Rightarrow \exists h, h' \in H, g_1^{-1}g_2h = h' \\
& \Rightarrow g_1^{-1}g_2 = h'h^{-1} \in H
\end{aligned}
$$

□

(e) → (d)

(a) $g_1 H = g_2 H$
(b) $H g_1^{-1} = H g_2^{-1}$
(c) $g_1 H \subseteq g_2 H$
(d) $g_2 \in g_1 H$
(e) $g_1^{-1} g_2 \in H$

$$
\begin{tikzcd}
\text{(a)} \arrow[dd] & & \text{(c)} \arrow[ll] \\
& \text{(b)} \arrow[dl, leftrightarrow] & \\
\text{(e)} \arrow[rr] & & \text{(d)} \arrow[uu]
\end{tikzcd}
$$

证明.

$$
\begin{aligned}
g_1^{-1} g_2 \in H \quad & \Rightarrow \exists h \in H, g_1^{-1} g_2 = h \\
& \Rightarrow \exists h \in H, g_2 = g_1 h \in g_1 H
\end{aligned}
$$

□

(d) → (c)

(a) $g_1H = g_2H$

(b) $Hg_1^{-1} = Hg_2^{-1}$

(c) $g_1H \subseteq g_2H$

(d) $g_2 \in g_1H$

(e) $g_1^{-1}g_2 \in H$

$$
\begin{tikzcd}
\text{(a)} \arrow[dd] & & \text{(c)} \arrow[ll] \\
& \text{(b)} & \\
\text{(e)} \arrow[rr] \arrow[ur, leftrightarrow] & & \text{(d)} \arrow[uu]
\end{tikzcd}
$$

证明.

$$
\begin{aligned}
g_2 \in g_1H \quad & \Rightarrow \exists h \in H, g_2 = g_1h \\
& \Rightarrow g_2h^{-1} = g_1 \\
& \Rightarrow \forall h' \in H, g_1h' = g_2h^{-1}h' \in g_2H \\
& \Rightarrow g_1H \subseteq g_2H
\end{aligned}
$$

□

(c) → (a)

(a) $g_1H = g_2H$
(b) $Hg_1^{-1} = Hg_2^{-1}$
(c) $g_1H \subseteq g_2H$
(d) $g_2 \in g_1H$
(e) $g_1^{-1}g_2 \in H$

$$\begin{tikzcd}
(a) \arrow[dd] & & (c) \arrow[ll] \\
& (b) & \\
(e) \arrow[ur, leftrightarrow] \arrow[rr] & & (d) \arrow[uu]
\end{tikzcd}$$

证明.

$$\begin{aligned}
g_1H \subseteq g_2H &\Rightarrow g_1 \in g_2H \\
&\overset{(d)\to(c)}{\Rightarrow} g_2H \subseteq g_1H \\
&\overset{g_1H\subseteq g_2H}{\Rightarrow} g_1H = g_2H
\end{aligned}$$

□

(e) → (b)

(a) $g_1H = g_2H$
(b) $Hg_1^{-1} = Hg_2^{-1}$
(c) $g_1H \subseteq g_2H$
(d) $g_2 \in g_1H$
(e) $g_1^{-1}g_2 \in H$

$$\begin{tikzcd}
\text{(a)} \arrow[dd] & & \text{(c)} \arrow[ll] \\
& \text{(b)} \arrow[dl] & \\
\text{(e)} \arrow[rr] & & \text{(d)} \arrow[uu]
\end{tikzcd}$$

证明.

$$\begin{aligned}
g_1^{-1}g_2 \in H \quad &\Rightarrow g_1^{-1} \in Hg_2^{-1} \\
&\Rightarrow \exists h \in H, g_1^{-1} = hg_2^{-1} \\
&\Rightarrow g_2^{-1} = h^{-1}g_1^{-1} \\
&\Rightarrow Hg_2^{-1} = Hh^{-1}g_1^{-1} = Hg_1^{-1}
\end{aligned}$$

□

(e) ← (b)

(a) $g_1 H = g_2 H$

(b) $H g_1^{-1} = H g_2^{-1}$

(c) $g_1 H \subseteq g_2 H$

(d) $g_2 \in g_1 H$

(e) $g_1^{-1} g_2 \in H$

$$\begin{tikzcd}
(a) \arrow[dd] & & (c) \arrow[ll] \\
& (b) & \\
(e) \arrow[ur] \arrow[rr] & & (d) \arrow[uu]
\end{tikzcd}$$

证明.

$$\begin{aligned} H g_2^{-1} = H g_1^{-1} &\Rightarrow H g_1^{-1} g_2 = H \\ &\Rightarrow g_1^{-1} g_2 \in H \end{aligned}$$

□

3-cycle

证明：A_n 中的每个置换皆可表成形如 $(k\ k+1\ k+2)$ 的 3-cycle 的乘积。

3-cycle

Let $p(k) = (k, k+1, k+2) = (k, k+1)(k+1, k+2)$

As A_n is finite, $\langle p(k) \rangle$ is finite.

Let $q(k) = p(k)^{-1} = (k+2, k+1)(k+1, k) = p(k)^z$, for some z

3-cycle

$\forall a, b, c \in G, (ac) = (ab)(bc)(ac)$, where a, b, c are different.

3-cycle

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$$\Downarrow$$

$$\begin{aligned}
(l, r) &= (l, l+1)(l+1, r)(l, l+1) \\
&= (l, l+1)(l+1, l+2)(l+2, r)(l+1, l+2)(l, l+1) \\
&\ldots \\
&= (l, l+1)(l+1, l+2)\cdots(r-2, r-1)(r-1, r)(r-2, r-1)\cdots(l+1, l+2)(l, l+1)
\end{aligned}$$

3-cycle

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\end{aligned}$$

Totally $2(r-l)-1$ transpositions.

3-cycle

$$(l, r) = (l, l+1)(l+1, l+2)\cdots(r-2, r-1)(r-1, r)(r-2, r-1)\cdots(l+1, l+2)(l, l+1)$$

3-cycle

$f(l, r)$: can be represented as a product of 3-cycles

$$(l, r) = \overline{(l, l+1)(l+1, l+2) \cdots (r-2, r-1)(r-1, r)(r-2, r-1) \cdots (l+1, l+2)}(l, l+1)$$

3-cycle

$f(l,r)$: can be represented as a product of 3-cycles

$$(l,r) = (l,l+1)(l+1,l+2)\cdots(r-2,r-1)(r-1,r)(r-2,r-1)\cdots(l+1,l+2)(l,l+1)$$

$g(l,r)$: can be represented as a product of 3-cycles

3-cycle

$f(l,r)$: can be represented as a product of 3-cycles

$$(l,r) = \overbrace{(l,l+1)(l+1,l+2)\cdots(r-2,r-1)(r-1,r)(r-2,r-1)\cdots(l+1,l+2)}^{f(l,r)}(l,l+1)$$

$$(l,r) = (l,l+1)\underbrace{(l+1,l+2)\cdots(r-2,r-1)(r-1,r)(r-2,r-1)\cdots(l+1,l+2)(l,l+1)}_{g(l,r)}$$

$g(l,r)$: can be represented as a product of 3-cycles

$$\boxed{(l,r) = (l,r)^{-1} = f(l,r)(l,l+1) = (l,l+1)g(l,r)}$$

3-cycle

Consider $(a, b)(c, d)$**, where** $a < b, c < d$

3-cycle

Consider $(a, b)(c, d)$**, where** $a < b, c < d$

$$(a, b)(c, d) = f(a, b)(a, a+1)(c, c+1)g(c, d)$$

3-cycle

$$\boxed{\text{Focus on } (a, a+1)(c, c+1)}$$

$$(1) a = c \quad (2) a < c \quad (3) a > c$$

3-cycle

$$\boxed{\text{Focus on } (a, a+1)(c, c+1)}$$

$$(1) a = c \;\; (2) a < c \;\; (3) a > c$$

- $a = c$

$$(a, a+1)(c, c+1) = e \Rightarrow (a, b)(c, d) = f(a, b)g(c, d).$$

3-cycle

3-cycle

- $a < c$

$$
\begin{aligned}
(a, a+1)(c, c+1) \quad &= p(a)(a+1, a+2)(c, c+1) \\
&= p(a)p(a+1)(a+2, a+3)(c, c+1) \\
&\cdots \\
&= p(a)p(a+1)\cdots p(c-2)(c-1, c)(c, c+1) \\
&= p(a)p(a+1)\cdots p(c-1)
\end{aligned}
.
$$

3-cycle

- $a < c$

$$
\begin{aligned}
(a, a+1)(c, c+1) &= p(a)(a+1, a+2)(c, c+1) \\
&= p(a)p(a+1)(a+2, a+3)(c, c+1) \\
&\cdots \\
&= p(a)p(a+1)\cdots p(c-2)(c-1, c)(c, c+1) \\
&= p(a)p(a+1)\cdots p(c-1)
\end{aligned}.
$$

- $a > c$

$$
\begin{aligned}
(a, a+1)(c, c+1) &= (a+1, a)(c+1, c) \\
&= q(a-1)(a, a-1)\cdots(c+1, c) \\
&= q(a-1)q(a-2)(a-1, a-2)\cdots(c+1, c) \\
&\cdots \\
&= q(a-1)q(a-2)\cdots(c+2, c+1)(c+1, c) \\
&= q(a-1)q(a-2)\cdots q(c)
\end{aligned}.
$$

3-cycle

$\boxed{\text{Therefore, } (a, a+1)(c, c+1) \text{ can be written as a product of 3-cycle of } (k, k+1, k+2)}$

3-cycle

$\boxed{\begin{array}{c}\text{Therefore, } (a, a+1)(c, c+1) \\ \text{can be written as a product of 3-cycle of} \\ (k, k+1, k+2)\end{array}}$

As each element $x \in A_n$ can be represented by even number of transpositions, x can be further written as a product of 3-cycle of $(k, k+1, k+2)$.


Thank
You!