作业1-8

UD第6章问题7、16、17

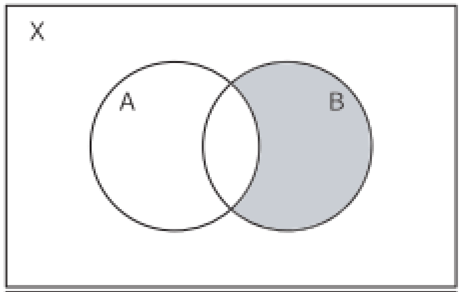
UD第7章问题1、8、9、10、11

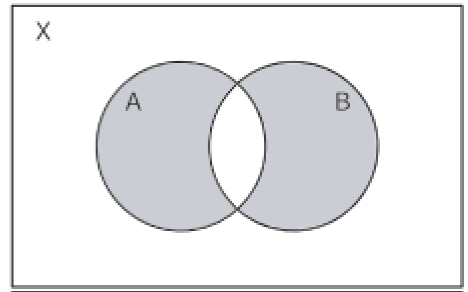
UD第8章问题1、4、7、8、9、11

UD第9章问题2、4、12、13、14、**16**

Find an expression for each of the shaded sets in the Venn diagrams of Figure 6.5.

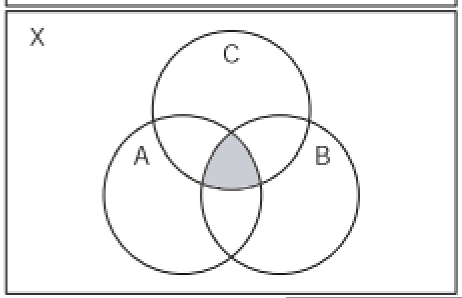
$B - A$

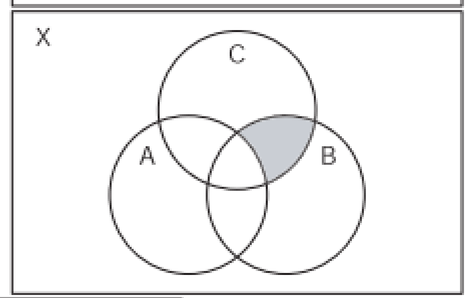




$(A \cup B) - (A \cap B)$

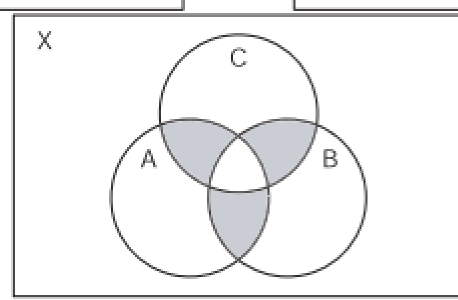
$A \cap B \cap C$





$(B \cup C) - A$

$(B \cup C) - (A \cap B \cap C)$



$$\big((A \cap B) \cup (A \cap C) \cup (B \cap C)\big)$$

$$-$$

$$\big((A \cup B) \cup (A \cup C) \cup (B \cup C)\big)$$

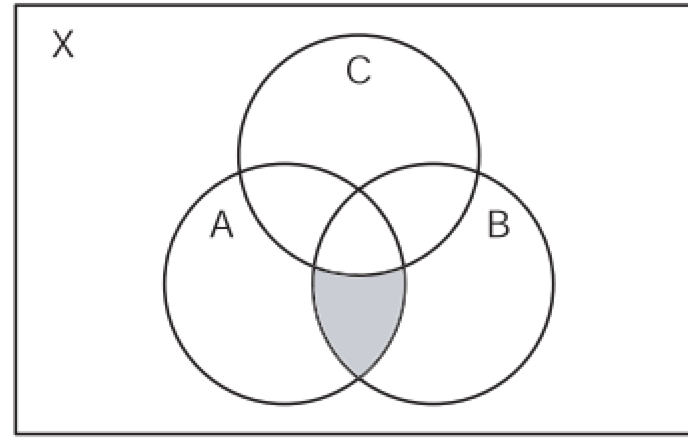
Problem 6.16. In each part of this problem, two sets, A and B, are defined. Prove that $A \subseteq B$ in each of the following:

(a) $A = \{x^2 : x \in \mathbb{Z}\}$ and $B = \mathbb{Z}$;
(b) $A = \mathbb{R}$ and $B = \{2x : x \in \mathbb{R}\}$;
(c) $A = \{(x,y) \in \mathbb{R}^2 : y = (5-3x)/2\}$ and $B = \{(x,y) \in \mathbb{R}^2 : 2y + 3x = 5\}$.

- (b) 目标$\forall x, (x \in A \Rightarrow x \in B)$
  - $x \in A \Rightarrow \exists y \in \mathbb{R} = A, x = 2y$
  - 所以$x \in B$

Problem 7.8. Consider the following sets:

(i) $(A \cap B) \setminus (A \cap B \cap C)$,
(ii) $A \cap B \setminus (A \cap B \cap C)$,
(iii) $A \cap B \cap C^c$,
(iv) $(A \cap B) \setminus C$, and
(v) $(A \setminus C) \cap (B \setminus C)$.



(a) Which of the sets above are written ambiguously, if any?
(b) Of the sets above that make sense, which ones equal the set sketched in Figure 7.2?
(c) Prove that $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$.

a) (ii)
- $(A \cap B) \backslash (A \cap B \cap C)$
- $A \cap (B \backslash (A \cap B \cap C))$

b) (i), (iii), (iv), (v)

c) $(A \cap B) \backslash C \equiv (A \cap B) \cap \neg C$
$\equiv (A \cap B) \cap (\neg C \cap \neg C)$
$\equiv (A \cap \neg C) \cap (B \cap \neg C)$
$\equiv (A \backslash C) \cap (B \backslash C)$

Problem 7.11. Prove or give a counterexample for the following statement.

Let X be the universe and $A, B \subseteq X$. If $A \cap Y = B \cap Y$ for **all** $Y \subseteq X$, then $A = B$.

7.11

$A \cap Y = B \cap Y$ for all $Y \subseteq X$.

$\Rightarrow (A \cap Y) - (B \cap Y) = \emptyset$

$\Rightarrow (A - B) \cap Y = \emptyset$

Since Y is not $\emptyset$ all the time, $A - B = \emptyset$.

$\Rightarrow$ A = B.

- 由于$X \subseteq X$，所以$A \cap X = B \cap X$
- 所以A=B

Problen 8.4 . Prove or give a counterexample: Let $\{A_n : n \in \mathbb{Z}^+\}$ and $\{B_n : n \in \mathbb{Z}^+\}$ be two indexed collections of sets. If $A_n \subset B_n$ for all $n \in \mathbb{Z}^+$, then

$$\bigcap_{n=1}^{\infty} A_n \subset \bigcap_{n=1}^{\infty} B_n.$$

Counterexample: If $A_k = B_m = \varnothing$ $(k \neq m)$, then $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} B_n = \varnothing$.

?

let $A_1=\{1,2\}$ $A_2=\{1,3\}$ $B_1=\{1,2,4\}$ $B_2=\{1,3,5\}$

满足对于任意 n ， $A_n \subset B_n$， 但是

$\bigcap_{n=1}^{\infty} A_n=\{1\}=\bigcap_{n=1}^{\infty} B_n=\{1\}$

Problem 8.8 Define

$$A = \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} \setminus \{-n, -n+1, \ldots, 0, \ldots, n-1, n\}).$$

The set A should be familiar to you. Guess what it is and then prove that your guess is correct.

- Guess: $A = \mathbb{Z}$
- Proof:
  - Let $X_n = \{-n, -n+1, \ldots, n-1, n\}, n \in \mathbb{Z}^+$
  - Let $B = \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} - \{-n, -n+1, \ldots, n-1, n\})$
    - $= \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} - X_n)$
    - $= \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} \cap \neg X_n)$
    - $= \mathbb{R} \cap \bigcap_{n \in \mathbb{Z}^+} (\neg X_n)$
    - $= \mathbb{R} \cap \neg\left(\bigcup_{n \in \mathbb{Z}^+} X_n\right)$
    - $= \mathbb{R} \cap \neg\mathbb{Z}$
  - $A = \mathbb{R} - B$
    - $= \mathbb{R} \cap \neg\mathbb{Z}$
    - $= \mathbb{R} \cap \neg(\mathbb{R} \cap \neg\mathbb{Z})$
    - $= \mathbb{R} \cap (\neg\mathbb{R} \cup \mathbb{Z})$
    - $= (\mathbb{R} \cap \neg\mathbb{R}) \cup (\mathbb{R} \cap \mathbb{Z})$
    - $\mathbb{Z}$

$$\left(\bigcup_{n \in \mathbb{Z}^+} X_n\right) = \mathbb{Z}$$

Problem[#] 8.11. A collection of sets $\mathscr{A}$ is said to be **pairwise disjoint** if the following is satisfied: For all $X, Y \in \mathscr{A}$, if $X \cap Y \neq \emptyset$, then $X = Y$.

A comment about this definition may be in order: Speaking informally, a collection of sets is pairwise disjoint if whenever we choose two sets from the collection, they are disjoint or they are equal.

(a) Give an example of a pairwise disjoint collection of infinitely many sets.
(b) What is the contrapositive of "if $X \cap Y \neq \emptyset$, then $X = Y$"?
(c) What is the converse of "if $X \cap Y \neq \emptyset$, then $X = Y$"?
(d) If $\mathscr{A}$ is a pairwise disjoint collection of sets, does the assertion you found in (b) hold for all $X, Y \in \mathscr{A}$?
(e) If the assertion that you found in (b) holds for all X and Y in some set $\mathscr{A}$, is $\mathscr{A}$ a pairwise disjoint collection of sets? **Yes, 直接同定义一致**
(f) Suppose that $\mathscr{B}$ is a pairwise disjoint collection of sets. Can we conclude that $\bigcap_{X \in \mathscr{B}} X = \emptyset$?
(g) Suppose that $\bigcap_{X \in \mathscr{B}} X = \emptyset$. Is $\mathscr{B}$ necessarily a pairwise disjoint collection of sets?

Problem# 8.11. A collection of sets $\mathscr{A}$ is said to be **pairwise disjoint** if the following is satisfied: For all $X, Y \in \mathscr{A}$, if $X \cap Y \neq \emptyset$, then $X = Y$.

A comment about this definition may be in order: Speaking informally, a collection of sets is pairwise disjoint if whenever we choose two sets from the collection, they are disjoint or they are equal.

(f) Suppose that $\mathscr{B}$ is a pairwise disjoint collection of sets. Can we conclude that $\bigcap_{X \in \mathscr{B}} X = \emptyset$?

- NO
  - Case1：$\mathcal{A}$仅包含一个集合
  - Case2：$\mathcal{A}$为multiset，且包含多个完全相同的集合。

Problem# 8.11. A collection of sets $\mathscr{A}$ is said to be **pairwise disjoint** if the following is satisfied: For all $X, Y \in \mathscr{A}$, if $X \cap Y \neq \emptyset$, then $X = Y$.

A comment about this definition may be in order: Speaking informally, a collection of sets is pairwise disjoint if whenever we choose two sets from the collection, they are disjoint or they are equal.

(g) Suppose that $\bigcap_{X \in \mathscr{B}} X = \emptyset$. Is $\mathscr{B}$ necessarily a pairwise disjoint collection of sets?

(g) No, If there is a set $A_\alpha = \varnothing$, $\bigcap_{\alpha \in I} A_\alpha = \varnothing$ whether $\{A_\alpha : \alpha \in I\}$ is a pairwise disjoint collection.

?

(g)no

设 $A_1=\{1,2\}, A_2=\{1,3\}, A_3=\{4\}$

这样满足条件但是却不是 pairwise disjoint collection

Problem 9.17. Let A, B, C, and D be nonempty sets. Then $A \times B = C \times D$ if and only if $A = C$ and $B = D$.

书上9.12

(a) Prove this statement.

(b) One of the two implications does not require the sets to be nonempty. Which one?

(c) If we do not require the sets to be nonempty, then the statement is false. Give examples of sets A, B, C, and D to show the necessity of the assumption that the sets be nonempty.

9.12

(a)

1. If A = C and B = D, then $A \times B = C \times D$.

2.If $A \times B = C \times D$.

For all $(x, y) \in A \times B$, it means $\begin{matrix} x \in A \\ y \in B \end{matrix}$

for all $x \in A, y \in B$, we have
$(x, y) \in A \times B = C \times D$
$\therefore x \in C, y \in D$

Since $A \times B = C \times D$, we get that $\begin{matrix} x \in C \\ y \in D \end{matrix}$. It is the same case when $(x, y) \in A \times B$.

So $\begin{matrix} A = C \\ B = D \end{matrix}$

Problem 9.23. This problem introduces rigorous definitions of an ordered pair and Cartesian product. Let A be a set and $a, b \in A$. We define the ordered pair of a and b with first coordinate a and second coordinate b as

$$(a,b) = \{\{a\},\{a,b\}\}.$$

Using this definition prove the following.

(a) If $(a,b) = (x,y)$, then $a = x$ and $b = y$.
(b) If $a \in A$ and $b \in B$, then $(a,b) \in \mathscr{P}(\mathscr{P}(A \cup B))$.

Now we are able to define the Cartesian product of the two sets A and B as the set

$$A \times B = \{x \in \mathscr{P}(\mathscr{P}(A \cup B)) : x = (a,b) \text{ for some } a \in A \text{ and some } b \in B\}.$$

(c) Using the definitions introduced in this problem, prove that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

This is a pretty complicated definition. It is also not our idea, but rather an idea that was born from axioms. P. Halmos's book, [41], is an excellent reference for this subject.

Problem 9.16. This problem introduces rigorous definitions of an ordered pair and Cartesian product. Let A be a set and $a, b \in A$. We define the ordered pair of a and b with first coordinate a and second coordinate b as

$$(a,b) = \{\{a\},\{a,b\}\}.$$

允许
multiset

Using this definition prove the following.

(a) If $(a,b) = (x,y)$, then $a = x$ and $b = y$.

先证a=x

$$\because (a,b) = (x,y)$$
$$\therefore \{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\}$$
$$又 \because 1 = |\{a\}| = |\{x\}| \neq |\{a,b\}| = |\{x,y\}| = 2$$
$$\therefore \{a\} = \{x\}, \{a,b\} = \{x,y\}$$
$$\therefore a = x$$

下证b=y,分情况讨论

case1： $a = b$, 可得x $= a = b = y$

case2： $a \neq b$,假设b$\neq y$

$$\because \{a,b\} = \{x,y\}$$
$$\therefore a = y, b = x$$
$$又 \because a = x$$
$$\therefore y = a = x = b$$

Problem 9.16. This problem introduces rigorous definitions of an ordered pair and Cartesian product. Let A be a set and $a, b \in A$. We define the ordered pair of a and b with first coordinate a and second coordinate b as

$$(a,b) = \{\{a\},\{a,b\}\}.$$

Using this definition prove the following.

(a) If $(a,b) = (x,y)$, then $a = x$ and $b = y$.

(b) If $a \in A$ and $b \in B$, then $(a,b) \in \mathscr{P}(\mathscr{P}(A \cup B))$.

$$\because a \in A, b \in B$$

$$\therefore a, b \in A \cup B$$

$$\therefore \{a\},\{a,b\} \subseteq A \cup B$$

$$\therefore \{a\},\{a,b\} \in \mathcal{P}(A \cup B)$$

$$\therefore (a,b) = \{\{a\},\{a,b\}\} \subseteq \mathcal{P}(A \cup B)$$

$$\therefore (a,b) = \{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$$

Now we are able to define the Cartesian product of the two sets A and B as the set

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(c) Using the definitions introduced in this problem, prove that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

This is a pretty complicated definition. It is also not our idea, but rather an idea that was born from axioms. P. Halmos's book, [41], is an excellent reference for this subject.

- 目标：$\forall x(x \in A \times B \Rightarrow x \in C \times D)$
- Proof：
  - Let $x = (a, b) = \{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$, for some $a \in A, b \in B$
  - $\because A \subseteq C, B \subseteq D$
  - $\therefore A \cup B \subseteq C \cup D$
  - $\therefore \mathcal{P}(\mathcal{P}(A \cup B)) \subseteq \mathcal{P}(\mathcal{P}(C \cup D))$
  - $\therefore x \in \mathcal{P}(\mathcal{P}(A \cup B))$ （1）
  - 又$\because a \in A \subseteq C, b \in B \subseteq D$ （2）
  - 由(1)(2)结合定义，可得 $x \in C \times D$

参考习题9.4结论：
$A \subseteq B$ iff $\mathcal{P}(A) \subseteq \mathcal{P}(B)$