- 教材讨论
  - JH第4章第1、2节

问题1：近似算法的基本概念

- 什么样的算法可以称作近似算法？

We start with the fundamental definition of approximation algorithms. Informally and roughly, an approximation algorithm for an optimization problem is an algorithm that provides a feasible solution whose quality does not differ too much from the quality of an optimal solution.

- 你怎么理解does not differ too much？

问题1：近似算法的基本概念 (续)

- 你怎么理解这些概念？
  - relative error

    $$\varepsilon_A(x) = \frac{|cost(A(x)) - Opt_U(x)|}{Opt_U(x)}.$$

    $$\varepsilon_A(n) = \max\{\varepsilon_A(x) \mid x \in L_I \cap (\Sigma_I)^n\}.$$

  - approximation ratio

    $$R_A(x) = \max\left\{\frac{cost(A(x))}{Opt_U(x)}, \frac{Opt_U(x)}{cost(A(x))}\right\}.$$

    $$R_A(n) = \max\{R_A(x) \mid x \in L_I \cap (\Sigma_I)^n\}.$$

  - δ-approximation algorithm

    $$R_A(x) \le \delta \ for\ every\ x \in L_I.$$

  - f(n)-approximation algorithm

    $$R_A(n) \le f(n)\ for\ every\ n \in \mathbb{N}.$$

问题1：近似算法的基本概念 (续)

- 这个算法的基本过程是什么？

Algorithm 4.2.1.3 (GMS (Greedy Makespan Schedule)).

Input: $I = (p_1, \ldots, p_n, m)$, n, m, $p_1, \ldots, p_n$ positive integers and $m \geq 2$.
Step 1: Sort $p_1, \ldots, p_n$.
To simplify the notation we assume $p_1 \geq p_2 \geq \cdots \geq p_n$ in the rest of the algorithm.
Step 2: **for** $i = 1$ **to** m **do**
  begin $T_i := \{i\}$;
    $Time(T_i) := p_i$
  end
{In the initialization step the m largest jobs are distributed to the m machines. At the end, T_i should contain the indices of all jobs assigned to the ith machine for $i = 1, \ldots, m$.}
Step 3: **for** $i = m + 1$ **to** n **do**
  begin compute an l such that
    $Time(T_l) := \min\{Time(T_j) | 1 \leq j \leq m\}$;
    $T_l := T_l \cup \{i\}$;
    $Time(T_l) := Time(T_l) + p_i$
  end
Output: $(T_1, T_2, \ldots, T_m)$.

问题1：近似算法的基本概念 (续)

- 你能逐步推导出它的approximation ratio吗？

$$Opt_{\mathrm{MS}}(I) \geq p_1 \geq p_2 \geq \cdots \geq p_n. \quad (4.1)$$

$$Opt_{\mathrm{MS}}(I) \geq \frac{\sum_{i=1}^{n} p_i}{m} \quad (4.2)$$

$$p_k \leq \frac{\sum_{i=1}^{k} p_i}{k} \quad (4.3)$$

(1) Let $n \leq m$.
Since $Opt_{\mathrm{MS}}(I) \geq p_1$ (4.1) and $cost(\{1\},\{2\},\ldots,\{n\},\emptyset,\ldots,\emptyset) = p_1$, GMS has found an optimal solution and so the approximation ratio is 1.

(2) Let $n > m$.
Let T_l be such that $cost(T_l) = \sum_{r \in T_l} p_r = cost(\mathrm{GMS}(I))$, and let k be the largest index in T_l. If $k \leq m$, then $|T_l| = 1$ and so $Opt_{\mathrm{MS}}(I) = p_1 = p_k$ and $\mathrm{GMS}(I)$ is an optimal solution.
Now, assume $m < k$. Following Figure 4.2 we see that

$$Opt_{\mathrm{MS}}(I) \geq cost(\mathrm{GMS}(I)) - p_k \quad (4.4)$$

because of $\sum_{i=1}^{k-1} p_i \geq m \cdot [cost(\mathrm{GMS}(I)) - p_k]$ and (4.2).

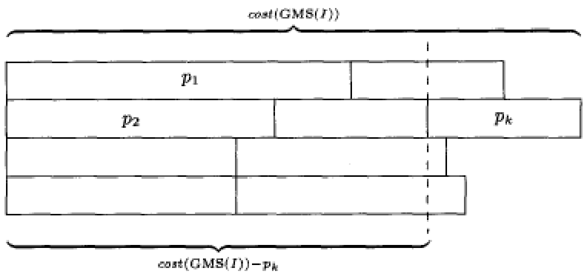

Fig. 4.2.

$$cost(\mathrm{GMS}(I)) - Opt_{\mathrm{MS}}(I) \underset{(4.4)}{\leq} p_k \underset{(4.3)}{\leq} \left(\sum_{i=1}^{k} p_i\right) \Big/ k. \quad (4.5)$$

$$\frac{cost(\mathrm{GMS}(I)) - Opt_{\mathrm{MS}}(I)}{Opt_{\mathrm{MS}}(I)} \underset{\substack{(4.5)\\(4.2)}}{\leq} \frac{\left(\sum_{i=1}^{k} p_i\right)/k}{\left(\sum_{i=1}^{n} p_i\right)/m} \leq \frac{m}{k} < 1.$$

问题1：近似算法的基本概念 (续)

- 你怎么理解PTAS和FPTAS？它们有什么区别？

Definition 4.2.1.6. *Let* $U = (\Sigma_I, \Sigma_O, L, L_I, \mathcal{M}, cost, goal)$ *be an optimization problem. An algorithm A is called a* **polynomial-time approximation scheme (PTAS) for** U, *if, for every input pair* $(x, \varepsilon) \in L_I \times \mathbb{R}^+$, *A computes a feasible solution* $A(x)$ *with a relative error at most* ε, *and* $Time_A(x, \varepsilon^{-1})$ *can be bounded by a function*[3] *that is polynomial in* $|x|$. *If* $Time_A(x, \varepsilon^{-1})$ *can be bounded by a function that is polynomial in both* $|x|$ *and* ε^{-1}, *then we say that A is a* **fully polynomial-time approximation scheme (FPTAS) for** U.

- 你怎么理解这两句话？
  - The advantage of PTASs is that the user has the choice of ε in this tradeoff of the quality of the output and the amount of computer work.
  - Probably a FPTAS is the best that one can have for a NP-hard optimization problem.

问题2：NPO的分类

- 你怎么理解这5种类型？分类的依据是什么？

NPO(I): Contains every optimization problem from NPO for which there exists a FPTAS.
{In Section 4.3 we show that the knapsack problem belongs to this class.}

NPO(II): Contains every optimization problem from NPO that has a PTAS.
{In Section 4.3.4 we show that the makespan scheduling problem belongs to this class.}

NPO(III): Contains every optimization problem $U \in$ NPO such that
(i) there is a polynomial-time δ-approximation algorithm for some $\delta > 1$, and
(ii) there is no polynomial-time d-approximation algorithm for U for some $d < \delta$ (possibly under some reasonable assumption like P $\neq$ NP), i.e., there is no PTAS for U.
{The minimum vertex cover problem, MAX-SAT, and $\triangle$-TSP are examples of members of this class.}

NPO(IV): Contains every $U \in NPO$ such that
(i) there is a polynomial-time $f(n)$-approximation algorithm for U for some $f : \mathbb{N} \to \mathbb{R}^+$, where f is bounded by a polylogarithmic function, and
(ii) under some reasonable assumption like P $\neq$ NP, there does not exist any polynomial-time δ-approximation algorithm for U for any $\delta \in \mathbb{R}^+$.
{The set cover problem belongs to this class.}

NPO(V): Contains every $U \in$ NPO such that if there exists a polynomial-time $f(n)$-approximation algorithm for U, then (under some reasonable assumption like P $\neq$ NP) $f(n)$ is not bounded by any polylogarithmic function.
{TSP and the maximum clique problem are well-known members of this class.}

问题3：stability

- 你觉得讨论stability的意义是什么？
- 你怎么理解这些概念？

Definition 4.2.3.1. *Let* $U = (\Sigma_I, \Sigma_O, L, L_I, \mathcal{M}, cost, goal)$ *and* $\overline{U} = (\Sigma_I, \Sigma_O, L, L, \mathcal{M}, cost, goal)$ *be two optimization problems with* $L_I \subset L$. *A* **distance function for** $\overline{U}$ **according to** L_I *is any function* $h_L : L \to \mathbb{R}^{\geq 0}$ *satisfying the properties*

(i) $h_L(x) = 0$ *for every* $x \in L_I$, *and*
(ii) h *is polynomial-time computable.*

Let h *be a distance function for* $\overline{U}$ *according to* L_I. *We define, for any* $r \in \mathbb{R}^+$,

$$\boldsymbol{Ball_{r,h}(L_I)} = \{w \in L \mid h(w) \leq r\}.^6$$

Let A *be a consistent algorithm for* $\overline{U}$, *and let* A *be an* ε*-approximation algorithm for* U *for some* $\varepsilon \in \mathbb{R}^{>1}$. *Let* p *be a positive real. We say that* A *is* **p-stable according to** $\boldsymbol{h}$ *if, for every real* $0 < r \leq p$, *there exists a* $\delta_{r,\varepsilon} \in \mathbb{R}^{>1}$ *such that* A *is a* $\delta_{r,\varepsilon}$*-approximation algorithm for* $U_r = (\Sigma_I, \Sigma_O, L, Ball_{r,h}(L_I), \mathcal{M}, cost, goal)$.

A is **stable according to h** *if A is p-stable according to h for every* $p \in \mathbb{R}^+$. *We say that A is* **unstable according to** $\boldsymbol{h}$ *if A is not p-stable for any* $p \in \mathbb{R}^+$.

For every positive integer r, *and every function* $f_r : \mathbb{N} \to \mathbb{R}^{>1}$ *we say that A is* $\boldsymbol{(r, f_r(n))}$**-quasistable according to h** *if A is an* $f_r(n)$*-approximation algorithm for* $U_r = (\Sigma_I, \Sigma_O, L, Ball_{r,h}(L_I), \mathcal{M}, cost, goal)$.

问题3：stability (续)

- 你怎么理解TSP中的这些distance？

$$dist(G,c) = \max\left\{0, \max\left\{\frac{c(\{u,v\})}{c(\{u,p\}) + c(\{p,v\})} - 1 \,\middle|\, u,v,p \in V(G),\ u \neq v, u \neq p, v \neq p\right\}\right\},$$

$$dist_k(G,c) = \max\left\{0, \max\left\{\frac{c(\{u,v\})}{\sum_{i=1}^{m} c(\{p_i,p_{i+1}\})} - 1 \,\middle|\, u,v \in V(G) \text{ and } u = p_1, p_2, \ldots, p_m = v \text{ is a simple path between } u \text{ and } v \text{ of length at most } k \text{ (i.e., } m+1 \leq k)\right\}\right\}$$

$$distance(G,c) = \max\{dist_k(G,c) \mid 2 \leq k \leq |V(G)| - 1\}.$$

- 例如，$Ball_{r,dist}(L_{\triangle})$ 中都是些什么？

$$c(\{u,v\}) \leq (1+r)(c(\{u,p\}) + c(\{p,v\}))$$

- 你能基于其它难问题，举出一些distance的例子吗？

问题3： stability (续)

- 你怎么理解PTAS的stability？

Note that applying the concept of stability to PTASs one can get two different outcomes. Let us consider a PTAS A as a collection of polynomial-time $(1+\varepsilon)$-approximation algorithms A_ε for every $\varepsilon \in \mathbb{R}^+$. If A_ε is stable according to a distance measure h for every $\varepsilon > 0$, then we can obtain either

(i) a PTAS for $U_r = (\Sigma_I, \Sigma_O, L, Ball_{r,h}(L_I), \mathcal{M}, cost, goal)$ for every $r \in \mathbb{R}^+$ (this happens, for instance, if $\delta_{r,\varepsilon} = 1 + \varepsilon \cdot f(r)$, where f is an arbitrary function), or

(ii) a $\delta_{r,\varepsilon}$-approximation algorithm for U_r for every $r \in \mathbb{R}^+$, but no PTAS for U_r for any $r \in \mathbb{R}^+$ (this happens, for instance, if $\delta_{r,\varepsilon} = 1 + r + \varepsilon$).

问题4：dual approximation

- 你觉得讨论dual approximation的意义是什么？
- 这里的distance和之前的distance有什么区别？
- 你怎么理解这些概念？

Definition 4.2.4.1. *Let $U = (\Sigma_I, \Sigma_O, L, L_I, \mathcal{M}, cost, goal)$ be an optimization problem. A* **constraint distance function** *for U is any function $h : L_I \times \Sigma_O^* \to \mathbb{R}^{\geq 0}$ such that*

(i) $h(x, S) = 0$ for every $S \in \mathcal{M}(x)$,
(ii) $h(x, S) > 0$ for every $S \notin \mathcal{M}(x)$, and
(iii) h is polynomial-time computable.

For every $\varepsilon \in \mathbb{R}^+$, and every $x \in L_I$, $\boldsymbol{\mathcal{M}_\varepsilon^h(x)}$ $= \{S \in \Sigma_O^* \mid h(x, S) \leq \varepsilon\}$ *is the* **ε-ball of $\mathcal{M}(x)$ according to h.**

Definition 4.2.4.2. *Let $U = (\Sigma_I, \Sigma_O, L, L_I, \mathcal{M}, cost, goal)$ be an optimization problem, and let h be a constraint distance function for U.*

An optimization algorithm A for U is called an **h-dual ε-approximation algorithm for U**, *if for every $x \in L_I$,*

(i) $A(x) \in \mathcal{M}_\varepsilon^h(x)$, and
(ii) $cost(A(x)) \geq Opt_U(x)$ if goal = maximum, and
$cost(A(x)) \leq Opt_U(x)$ if goal = minimum.

Definition 4.2.4.3. *Let $U = (\Sigma_I, \Sigma_O, L, L_I, \mathcal{M}, cost, goal)$ be an optimization problem, and let h be a constraint distance function for U.*

An algorithm A is called **h-dual polynomial-time approximation scheme (h-dual PTAS for U)**, *if*

(i) for every input $(x, \varepsilon) \in L_I \times \mathbb{R}^+$, $A(x, \varepsilon) \in \mathcal{M}_\varepsilon^h(x)$,
(ii) $cost(A(x, \varepsilon)) \geq Opt_U(x)$ if goal = maximum, and
$cost(A(x, \varepsilon)) \leq Opt_U(x)$ if goal = minimum, and
(iii) $Time_A(x, \varepsilon^{-1})$ is bounded by a function that is polynomial in $|x|$.

If $Time_A(x, \varepsilon^{-1})$ can be bounded by a function that is polynomial in both $|x|$ and ε^{-1}, then we say that A is a **h-dual fully polynomial-time approximation scheme (h-dual FPTAS) for U.**

问题4：dual approximation (续)

- 你怎么理解BIN-P中的这个distance？

$$h(I,T) = \max\left\{0, \max\left\{\sum_{l\in T_i} p_l \,\middle|\, i = 1, 2, \ldots, m\right\} - 1\right\}.$$

- 你能基于其它难问题，举出一些distance的例子吗？