

- 教材讨论  
– TC第28章

# 问题1：线性方程组求解

- 如何将线性方程组表示为矩阵形式？
- LUP分解的矩阵表示是什么？
  - L、U、P分别是怎样的矩阵？
- 如何用它来改写线性方程组的矩阵表示？

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n. \end{aligned}$$

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$$Ax = b$$

$$PA = LU$$



$$LUX = Pb$$

L: unit lower-triangular matrix

U: upper-triangular matrix

P: permutation matrix

# 问题1：线性方程组求解 (续)

- 接下来如何分两步求解？  $LUx = Pb$
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$$LUx = Pb$$



$$y = Ux$$

$$Ly = Pb$$



$$Ux = y$$

# 问题1: 线性方程组求解 (续)

- 你理解LUP-SOLVE了吗?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0.2 & 1 & 0 \\ 0.6 & 0.5 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 6 & 3 \\ 0 & 0.8 & -0.6 \\ 0 & 0 & 2.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 1.4 \\ 1.5 \end{pmatrix}$$

$$\begin{aligned} y_1 &= b_{\pi[1]}, & u_{11}x_1 + u_{12}x_2 + \cdots + u_{1,n-2}x_{n-2} + u_{1,n-1}x_{n-1} + u_{1n}x_n &= y_1, \\ l_{21}y_1 + y_2 &= b_{\pi[2]}, & u_{22}x_2 + \cdots + u_{2,n-2}x_{n-2} + u_{2,n-1}x_{n-1} + u_{2n}x_n &= y_2, \\ l_{31}y_1 + l_{32}y_2 + y_3 &= b_{\pi[3]}, & & \vdots \\ &\vdots & & \\ l_{n1}y_1 + l_{n2}y_2 + l_{n3}y_3 + \cdots + y_n &= b_{\pi[n]}, & u_{n-2,n-2}x_{n-2} + u_{n-2,n-1}x_{n-1} + u_{n-2,n}x_n &= y_{n-2}, \\ & & u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n &= y_{n-1}, \\ & & u_{n,n}x_n &= y_n. \end{aligned}$$

LUP-SOLVE( $L, U, \pi, b$ )

```

1  n = L.rows
2  let x and y be new vectors of length n
3  for i = 1 to n
4      y_i = b_{\pi[i]} - \sum_{j=1}^{i-1} l_{ij}y_j
5  for i = n downto 1
6      x_i = (y_i - \sum_{j=i+1}^n u_{ij}x_j) / u_{ii}
7  return x

```

# 问题1: 线性方程组求解 (续)

- 你理解LU-DECOMPOSITION了吗?

$$\begin{aligned}
 A &= \begin{pmatrix} a_{11} & w^T \\ v & A' \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 \\ v/a_{11} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{11} & w^T \\ 0 & A' - vw^T/a_{11} \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 \\ v/a_{11} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{11} & w^T \\ 0 & L'U' \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 \\ v/a_{11} & L' \end{pmatrix} \begin{pmatrix} a_{11} & w^T \\ 0 & U' \end{pmatrix} \\
 &= LU,
 \end{aligned}$$

LU-DECOMPOSITION(A)

```

1  n = A.rows
2  let L and U be new n x n matrices
3  initialize U with 0s below the diagonal
4  initialize L with 1s on the diagonal and 0s above the diagonal
5  for k = 1 to n
6      ukk = akk
7      for i = k + 1 to n
8          lik = aik/akk      // aik holds vi
9          uki = aki      // aki holds wi
10     for i = k + 1 to n
11         for j = k + 1 to n
12             aij = aij - likukj
13  return L and U
    
```

2 3 1 5  
6 13 5 19  
2 19 10 23  
4 10 11 31

(a)

<b>2</b>	3	1	5
3	4	2	4
1	16	9	18
2	4	9	21

(b)

2	3	1	5
3	<b>4</b>	2	4
1	4	1	2
2	1	7	17

(c)

2	3	1	5
3	4	2	4
1	4	<b>1</b>	2
2	1	7	3

(d)

# 问题1: 线性方程组求解 (续)

- 为什么要permutation?
- 你能解释这些步骤吗?

$$\begin{aligned}QA &= \begin{pmatrix} a_{k1} & w^T \\ v & A' \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ v/a_{k1} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{k1} & w^T \\ 0 & A' - vw^T/a_{k1} \end{pmatrix}\end{aligned}$$

$$P'(A' - vw^T/a_{k1}) = L'U'$$

$$P = \begin{pmatrix} 1 & 0 \\ 0 & P' \end{pmatrix} Q$$

$$\begin{aligned}PA &= \begin{pmatrix} 1 & 0 \\ 0 & P' \end{pmatrix} QA \\ &= \begin{pmatrix} 1 & 0 \\ 0 & P' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v/a_{k1} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{k1} & w^T \\ 0 & A' - vw^T/a_{k1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ P'v/a_{k1} & P' \end{pmatrix} \begin{pmatrix} a_{k1} & w^T \\ 0 & A' - vw^T/a_{k1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ P'v/a_{k1} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{k1} & w^T \\ 0 & P'(A' - vw^T/a_{k1}) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ P'v/a_{k1} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{k1} & w^T \\ 0 & L'U' \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ P'v/a_{k1} & L' \end{pmatrix} \begin{pmatrix} a_{k1} & w^T \\ 0 & U' \end{pmatrix} \\ &= LU,\end{aligned}$$



# 问题2： 矩阵求逆

- 你能简要描述利用LUP分解求逆矩阵的思路吗？
- The proof of Theorem 28.2 suggests a means of solving the equation  $Ax=b$  by using LU decomposition **without pivoting**, so long as  $A$  is nonsingular.  
你理解这种新方法了吗？

# 问题2： 矩阵求逆

- 你能简要描述利用LUP分解求逆矩阵的思路吗？

$$AX = I_n$$

$$AX_i = e_i$$

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你理解这种新方法了吗？

1.  $(A^T A)x = A^T b$ .
2. Factor the symmetric positive-definite matrix  $A^T A$  by computing an LU decomposition.
3. Use forward and back substitution to solve for  $x$  with the right-hand side  $A^T b$ .

# 问题3：求行列式

- 我们还可以利用LUP分解来求方阵的行列式，你能想到吗？

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- 我们还可以利用LUP分解来求方阵的行列式，你能想到吗？

$$A = P^{-1}LU$$

$$\det(A) = \det(P^{-1}) \det(L) \det(U) = (-1)^S \left( \prod_{i=1}^n l_{ii} \right) \left( \prod_{i=1}^n u_{ii} \right).$$

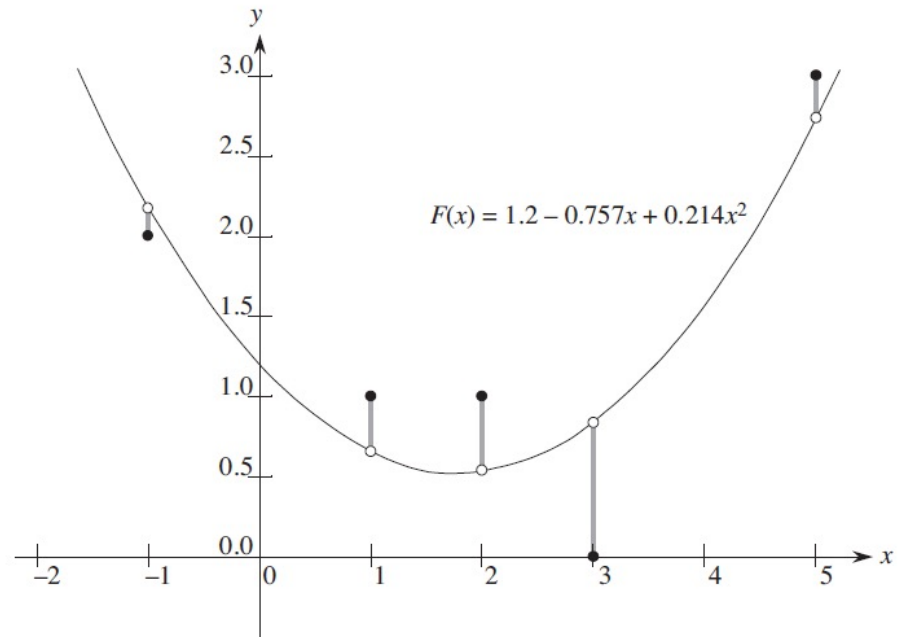
# 问题4：最小二乘法

- 最小二乘法想要解决的是一个什么问题？

$(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)$

$$\eta_i = F(x_i) - y_i$$

$$\|\eta\| = \left( \sum_{i=1}^m \eta_i^2 \right)^{1/2}$$



# 问题4：最小二乘法 (续)

- 你能解释这些步骤吗？

$$\|\eta\| = \left( \sum_{i=1}^m \eta_i^2 \right)^{1/2}$$

$$\|\eta\|^2 = \|Ac - y\|^2 = \sum_{i=1}^m \left( \sum_{j=1}^n a_{ij}c_j - y_i \right)^2$$

$$\frac{d \|\eta\|^2}{dc_k} = \sum_{i=1}^m 2 \left( \sum_{j=1}^n a_{ij}c_j - y_i \right) a_{ik} = 0$$

$$(Ac - y)^T A = 0$$

$$A^T(Ac - y) = 0$$

$$A^T Ac = A^T y$$

$$c = ((A^T A)^{-1} A^T) y$$