Counterfeit Coin Problems

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In January of 1945, the following problem appeared in the *American Mathematical Monthly*, contributed by E. D. Schell:

> You have eight similar coins and a beam balance. At most one coin is counterfeit and hence underweight. How can you detect whether there is an underweight coin, and if so, which one, using the balance only twice?

Since such weighing problems are today as much a part of the tradition of recreational mathematics as magic squares and mobius bands, it is interesting to note that they date only from that problem in 1945. The classic works of Loyd, Ball, Dudeney and Kraitchik contain no such problems. The responses to Schell's problem, a flurry of papers in the *Monthly*, *Scripta Mathematica*, and the *Mathematical Gazette*, contain no mention of earlier publications which might be relevant. Thus, it is apparent that this class of extremely natural and appealing puzzles is a recent invention, not an old chestnut which "crops up from time to time to puzzle and infuriate new generations of solvers" as someone wrote in 1961 (when such puzzles were just fifteen years old!).

That early spate of papers, appearing in 1945 and the next few years with a speed unheard of in these days of publication backlogs, solved, resolved, and generalized the original problem in all directions. In this paper I present several variations of the balancing problem, all of which have been solved before. The method of solution given here may be original.

We will always be dealing with a set of coins, of identical appearance, and a beam balance. We are interested in minimizing the maximum number of weighings which may be required to find the odd coin. The method of solution chosen may require solution of several types of weighing problems simultaneously, because a problem may change character after a weighing. For example, after a single use of the beam we have some coins which we know to be genuine (those on the beam, if it balances, those left off it if it does not). Thus, after one weighing we have a problem of a different type than the original one.

We shall break all of the problems dealt with into two classes: those in which the counterfeit coin is known to be underweight and those in which it is only known to be of a different weight than the genuine coins. At the very end of the paper we will examine the possibility of no counterfeit coin. For the time being, we will assume that in every case exactly one coin is not genuine. Sometimes a "standard" coin, known to be the correct weight, will be provided. We begin with the first class of problems.

Theorem 1. *Let S be a set of coins, one lighter than the rest. The least number of weighings on a beam balance in which the light coin can be found is the unique n satisfying* $3^{n-1} < |S| \leqq 3^n$.

Proof. Suppose first that $|S| = 3^n$. Then for the first weighing, we divide S into three equal sets, S_1, S_2 and S_3, of size 3^{n-1}, and place S_1 and S_2 on opposite sides of the beam. If the scale does not balance, the light coin is on the light side of the scale; otherwise it is in S_3. In any case, we have reduced our problem to finding the light coin in a set of size 3^{n-1}. Continuing in this way, the light coin can be located after n weighings. If $|S| < 3^n$, a similar procedure can be followed, placing equal sets S_1 and S_2 of coins on the scale, leaving a set S_3 of at most 3^{n-1} coins unweighed. Again, repetitions will lead us to the light coin in at most $n-1$ more steps.

On the other hand, it is clear that a single weighing of any sort cannot do better than cut the size of the set of "suspect" coins by a factor of 3. This is so because three sets are involved in the process, two on the scale and one off, and the outcome merely distinguishes which of the three sets contains the light coin. Thus if $|S| > 3^{n-1}$, $n-1$ weighings cannot be enough to find the light coin in all cases.

We now modify the problem under consideration by assuming only that the counterfeit coin is a different weight from the others — either heavier or lighter. The general observation we need to solve this more difficult problem is the following rather strange sounding result.

LEMMA. *If in a set S of coins one coin is a different weight than the rest and each coin is labelled "possibly heavy"* (p.h.) *or "possibly light"* (p.l.), *the least number of weighings on a beam balance in which the odd coin can be found is the unique n satisfying* $3^{n-1} < |S| \leqq 3^n$.

Proof. Notice that this result is very similar to Theorem 1, in which every coin can be thought of as being labelled p.l. since the odd coin is known to be light. In fact, the weighing procedure of Theorem 1 works in this case, with one restriction. Whenever we place coins on the scale, we must be sure to put equal numbers of p.l. coins on the two sides (and therefore equal numbers of p.h. coins on the two sides, as well). If, for example, $|S| = 3^n$, we divide S into three sets of size 3^{n-1}, say S_1, S_2 and S_3, placing the same number of p.l. coins in S_1 and S_2. When S_1 and S_2 are compared on the scale, if S_1 (say) is heavier, the counterfeit coin must be among the p.h. coins in S_1 or the p.l. coins in S_2, which together constitute a set of size 3^{n-1}. If the scale balances, we are, of course, left with the set S_3, of size 3^{n-1}. Thus in every case we reduce the size of the set by a factor of 3, as in Theorem 1. For sets where $|S|$ is not a power of 3 a similar procedure is effective.

The lemma is useful because it describes a common type of weighing problem. If we know only that the counterfeit coin is an odd weight, a first weighing which does not balance leaves us with the coins on the scale each labelled "possibly heavy" or "possibly light". We are now ready to solve the problem of the odd coin in the case where a standard coin is provided.

THEOREM 2. *If we are given a set S of coins, plus a standard coin, and one coin in S is a different weight than the rest, then the least number of weighings in which the odd coin can be found is the unique n satisfying* $(3^{n-1}-1)/2 < |S| \leqq (3^n - 1)/2$.

Proof. Let us denote by $M(n)$ the maximum number of coins for which the odd coin problem can be solved in n weighings if a standard coin is provided. The lemma claims that $M(n) = (3^n - 1)/2$. It is easy to see that this is correct for $n = 1$ and 2.

Suppose now we are given a set S of coins from which we are to find the odd coin in n weighings. On our first weighing, we must place equal sets of coins S_1 and S_2 on the scale, leaving off a set S_3. If the beam balances, we are left with S_3, so we must require $|S_3| \leqq M(n-1)$ to be able to solve the problem in that case. On the other hand, if the scale does not balance, we are left with S_1 and S_2, each coin labelled "possibly heavy" or "possibly light". So we must have $|S_1| + |S_2| \leqq 3^{n-1}$. Since the two sides of the scale must have the same number of coins, this apparently implies that $|S_1| + |S_2| = 3^{n-1} - 1$, at a maximum, but that is not so. We have a standard coin at our disposal, so we can let $|S_1| + |S_2| = 3^{n-1}$, and $|S_1| = |S_2| + 1$, using the standard coin on the S_2 side. So if the scale does not balance, we are left with 3^{n-1} coins, each labelled, a problem we know by the lemma to be solvable.

Thus we have found that we can solve the balancing problem if $|S| = (|S_1| + |S_2|) + |S_3| = 3^{n-1} + M(n-1)$. This yields $M(n) = 3^{n-1} + M(n-1)$, which leads to $M(n) = \sum_{i=0}^{n-1} 3^i$. Thus $M(n) = (3^n - 1)/2$, as the sum of a geometric series.

THEOREM 3. *Let S be a set of more than two coins, one a different weight than the others. The least number of weighings in which the odd coin can be found is the unique n satisfying* $(3^{n-1}-3)/2 < |S| \leqq (3^n - 3)/2$.

Proof. Clearly we must begin by comparing two equal sets S_1 and S_2 on the balance, leaving off a set S_3. If the beam does not balance, the counterfeit is in $S_1 \cup S_2$, each coin is labelled "possibly heavy" or "possibly light" and so, by the lemma, $|S_1 \cup S_2| \leqq 3^{n-1}$. Since we must balance equal sets of coins and have no standard coin, the maximum for $|S_1 \cup S_2|$ is actually $3^{n-1} - 1$. If, on the other hand, the beam balances, we are left with S_3 and some standard coins (in S_1 and S_2). Thus, by Theorem 2,

$|S_3|$ can be as large as $(3^{n-1}-1)/2$. Combining these results, we find $|S|$ can be $(3^{n-1}-1)+(3^{n-1}-1)/2=(3^n-3)/2$, as desired.

Theorem 1 shows that an underweight coin in a set of k coins can be found in $\lceil \log_3 k \rceil$ weighings where $\lceil x \rceil$ is the least integer greater than or equal to x. As can easily be seen, a standard coin (or coins) does not reduce that number. If we know only that the counterfeit coin is a different weight, then $\lceil \log_3(2k+3) \rceil$ weighings are required, as proved in Theorem 3. In that case, however, a standard coin is a help occasionally, since Theorem 2 states that $\lceil \log_3(2k+1) \rceil$ weighings are required if a standard coin is provided.

None of this actually deals with Schell's original problem, which presents us with a set of coins which *may* contain a counterfeit one or may not. We have assumed that exactly one counterfeit was present in every problem. It is easy to see, by following through the process outlined in Theorem 1, that if the beam balances on every weighing, there is exactly one coin which has never been on the scale. If we are not sure whether or not we have a counterfeit coin, that last coin is an embarrassment! Thus, exactly one fewer coins, (3^n-1), can be handled in n weighings in that case. In the procedure outlined for an "odd" coin, there is no such problem, because if all of the coins are the same weight, they will all eventually be placed on the scale.

References

Many of the early references to balancing problems were in sections devoted to problems or mathematical notes, and were untitled. We therefore list some of them in an informal way, by journal, indicating the page number and author of each article. The author is indebted to the referee for pointing out many of these references.

American Mathematical Monthly: 1945 (42, E. D. Schell; 397, M. Dernham), 1946 (156, D. Eves; 278, N. J. Fine), 1947 (46, E. D. Schell; 48, J. Rosenbaum).

Mathematical Gazette: 1945 (227, R. L. Goodstein), 1946 (231, F. S. Dyson), 1947 (31, C. A. B. Smith).

Scripta Mathematica: 1945 (360, H. D. Grossman; 361, L. Withington, Jr.), 1948 (66, C. W. Raine; 67, K. Itkin; 69, H. D. Grossman).

Geoboard Triangles with One Interior Point

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A geoboard is a physical model of an integer lattice, the set of points in the plane with integer coordinates. Nails are hammered in a square array on a piece of wood and rubber bands are stretched over the nails to make polygons. A central formula of geoboard geometry is Pick's formula for the area of a polygon with vertices on the lattice: Area $= T/2+(I-1)$ where T is the number of points at which the polygon intersects the lattice and I is the number of lattice points enclosed by, but not touching, the polygon. (See [**1**], pp. 208–209.)

It is natural to ask what combinations of T and I can occur in polygons. For example, if $I=0$, T can assume any value greater than 2. This can be seen by making a triangle that touches $T-1$ points on the x-axis and with third vertex on the line $y=1$. In contrast to this is the case $I=1$. A well-known problem states, for example, that there is no geoboard triangle touching seven lattice points and enclosing exactly one. Proofs of this fact bring together elementary geometry and number theory and reveal some of the reasons that geoboards can be fascinating. In this note we will prove a more general theorem that includes the above problem as a special case.