3-10 Conectivity

Jun Ma

majun@nju.edu.cn

November 25, 2020

CZ 5.4

Prove that if v is a cut-vertex of a graph G, then v is **not** a cut-vertex of the **complement** $\bar{G}$ of G.

CZ 5.4

Prove that if v is a cut-vertex of a graph G, then v is **not** a cut-vertex of the **complement** $\bar{G}$ of G.

Proof.

- As v is a cut-vertex of G, $G - v$ has at least two different components. Let $C = \{C_1, C_2, ..., C_m\}$ be the set of all components of $G - v$

CZ 5.4

Prove that if v is a cut-vertex of a graph G, then v is **not** a cut-vertex of the **complement** $\bar{G}$ of G.

Proof.

- As v is a cut-vertex of G, $G - v$ has at least two different components. Let $C = \{C_1, C_2, ..., C_m\}$ be the set of all components of $G - v$
- Now consider any two different vertex $u, w \in \bar{G}$

CZ 5.4

Prove that if v is a cut-vertex of a graph G, then v is **not** a cut-vertex of the **complement** $\bar{G}$ of G.

Proof.

- As v is a cut-vertex of G, $G - v$ has at least two different components. Let $C = \{C_1, C_2, ..., C_m\}$ be the set of all components of $G - v$
- Now consider any two different vertex $u, w \in \bar{G}$
  - **Case 1 ($u \in C_i, w \in C_j$ for some $i \neq j$):** $(u, w) \in (\bar{G} - v).E$

CZ 5.4

Prove that if v is a cut-vertex of a graph G, then v is **not** a cut-vertex of the **complement** $\bar{G}$ of G.

Proof.

- As v is a cut-vertex of G, $G - v$ has at least two different components. Let $C = \{C_1, C_2, ..., C_m\}$ be the set of all components of $G - v$
- Now consider any two different vertex $u, w \in \bar{G}$
  - **Case 1 ($u \in C_i, w \in C_j$ for some $i \neq j$):** $(u, w) \in (\bar{G} - v).E$
  - **Case 2 ($u, w \in C_i$ for some i):** there must be at least one vertex $x \in C_j (i \neq j)$, s.t. $(u, x) \in (\bar{G} - v).E$ and $(w, x) \in (\bar{G} - v).E$. So, u and w are connected in $(\bar{G} - v)$

CZ 5.4

Prove that if v is a cut-vertex of a graph G, then v is **not** a cut-vertex of the **complement** $\bar{G}$ of G.

Proof.

- As v is a cut-vertex of G, $G - v$ has at least two different components. Let $C = \{C_1, C_2, ..., C_m\}$ be the set of all components of $G - v$
- Now consider any two different vertex $u, w \in \bar{G}$
  - **Case 1** ($u \in C_i, w \in C_j$ **for some** $i \neq j$)**:** $(u, w) \in (\bar{G} - v).E$
  - **Case 2** ($u, w \in C_i$ **for some** i)**:** there must be at least one vertex $x \in C_j (i \neq j)$, s.t. $(u, x) \in (\bar{G} - v).E$ and $(w, x) \in (\bar{G} - v).E$. So, u and w are connected in $(\bar{G} - v)$
- $\bar{G} - v$ is connected and v is not a cut-vertex of $\bar{G}$

□

CZ 5.8 (a)

Let G be a nontrivial connected graph. Prove that if v is an **end-vertex** of a **spanning tree** of G, then v is **not** a cut-vertex of G.

CZ 5.8 (a)

Let G be a nontrivial connected graph. Prove that if v is an **end-vertex** of a **spanning tree** of G, then v is **not** a cut-vertex of G.

Proof.

- Let T be a spanning tree of G, in which v is an end-vertex.

CZ 5.8 (a)

Let G be a nontrivial connected graph. Prove that if v is an **end-vertex** of a **spanning tree** of G, then v is **not** a cut-vertex of G.

Proof.

- Let T be a spanning tree of G, in which v is an end-vertex.
- Then, $T - v$ is also a tree connecting all vertices other than v

CZ 5.8 (a)

Let G be a nontrivial connected graph. Prove that if v is an **end-vertex** of a **spanning tree** of G, then v is **not** a cut-vertex of G.

Proof.

- Let T be a spanning tree of G, in which v is an end-vertex.
- Then, $T - v$ is also a tree connecting all vertices other than v
- So $G - v$ is connected and v cannot be a cut-vertex.

□

CZ 5.8 (b)

Use (a) to give an alternative proof of the fact that every nontrivial connected graph contains at least two vertices that are not cut-vertices.

CZ 5.8 (b)

Use (a) to give an alternative proof of the fact that every nontrivial connected graph contains at least two vertices that are not cut-vertices.

Proof.

- Let T be a spanning tree of G, there must be at least two different vertices $u, v \in G.V$, s.t. u and v are end-vertices of T.

CZ 5.8 (b)

Use (a) to give an alternative proof of the fact that every nontrivial connected graph contains at least two vertices that are not cut-vertices.

Proof.

- Let T be a spanning tree of G, there must be at least two different vertices $u, v \in G.V$, s.t. u and v are end-vertices of T.
- According to (a), u and v cannot be cut-vertices.

□

CZ 5.8 (c)

Let v be a vertex in a nontrivial connected graph G. Show that there exists a spanning tree of G that contains all edges of G that are incident with v.

CZ 5.8 (c)

Let v be a vertex in a nontrivial connected graph G. Show that there exists a spanning tree of G that contains all edges of G that are incident with v.

Proof.

- Let T be a spanning tree of G and assume that the edge $(v, u) \in G.E$ does not exist in T

CZ 5.8 (c)

Let v be a vertex in a nontrivial connected graph G. Show that there exists a spanning tree of G that contains all edges of G that are incident with v.

Proof.

- Let T be a spanning tree of G and assume that the edge $(v, u) \in G.E$ does not exist in T
- There must be a path $P : v \rightsquigarrow u$ in T

CZ 5.8 (c)

Let v be a vertex in a nontrivial connected graph G. Show that there exists a spanning tree of G that contains all edges of G that are incident with v.

Proof.

- Let T be a spanning tree of G and assume that the edge $(v, u) \in G.E$ does not exist in T
- There must be a path $P : v \rightsquigarrow u$ in T
- Remove any edge within P from T and add the edge (v, u) to T would yield a new spanning tree T' of G that contains (v, u)

CZ 5.8 (c)

Let v be a vertex in a nontrivial connected graph G. Show that there exists a spanning tree of G that contains all edges of G that are incident with v.

Proof.

- Let T be a spanning tree of G and assume that the edge $(v, u) \in G.E$ does not exist in T
- There must be a path $P : v \rightsquigarrow u$ in T
- Remove any edge within P from T and add the edge (v, u) to T would yield a new spanning tree T' of G that contains (v, u)
- Repeat the process we could obtain a spanning tree T^* containing all edges of G that are incident with v.

□

CZ 5.8 (d)

Prove that if a connected graph G has exactly two vertices that are not cut-vertices, then G is a path.

Proof.

CZ 5.8 (d)

Prove that if a connected graph G has exactly two vertices that are not cut-vertices, then G is a path.

Proof.

- Part-1 (there is no vertex whose degree is greater than 2.)

- Part-2 (G has exactly two vertices with degree 1.)

- So, G is a **connected** graph with exactly two vertices of **degree 1** and all other vertices of **degree 2**.

□

CZ 5.8 (d)

Prove that if a connected graph G has exactly two vertices that are not cut-vertices, then G is a path.

Proof.

- Part-1 (there is no vertex whose degree is greater than 2.)
  - Assume that there is a vertex $v \in G$, s.t. $deg(v) >= 3$
  - According to (c), there must be a spanning tree T containing all edges that are incident with v.
  - Then, T must have more than two end-vertices which are not cut-vertices of G as well. **Conflict**!
- Part-2 (G has exactly two vertices with degree 1.)
- So, G is a **connected** graph with exactly two vertices of **degree 1** and all other vertices of **degree 2**.

□

CZ 5.8 (d)

Prove that if a connected graph G has exactly two vertices that are not cut-vertices, then G is a path.

Proof.

- Part-1 (there is no vertex whose degree is greater than 2.)
  - Assume that there is a vertex $v \in G$, s.t. $deg(v) >= 3$
  - According to (c), there must be a spanning tree T containing all edges that are incident with v.
  - Then, T must have more than two end-vertices which are not cut-vertices of G as well. **Conflict**!
- Part-2 (G has exactly two vertices with degree 1.)
  - Assume there are at least 3 vertices whose degree are equal to 1.
  - Then these vertices cannot be cut-vertices. **Conflict**!
- So, G is a **connected** graph with exactly two vertices of **degree 1** and all other vertices of **degree 2**.

CZ 5.10

Prove that a connected graph G of size at least 2 is non-separable if and only if any two adjacent edges of G lie on a common cycle of G.

$\Leftarrow$

Proof.

$\Leftarrow$

Proof.

- Assume v is a cut-vertex of G, and let $M = \{c_1, c_2, ..., c_m\}$ be the set of components of $G - v$

$\Leftarrow$

Proof.

- Assume v is a cut-vertex of G, and let $M = \{c_1, c_2, ..., c_m\}$ be the set of components of $G - v$
- Let $u \in c_i \cap N(v)$ and $w \in c_j \cap N(v)$, where $N(v)$ is the neighbor set of v and $i \neq j$

$\Leftarrow$

Proof.

- Assume v is a cut-vertex of G, and let $M = \{c_1, c_2, ..., c_m\}$ be the set of components of $G - v$
- Let $u \in c_i \cap N(v)$ and $w \in c_j \cap N(v)$, where $N(v)$ is the neighbor set of v and $i \neq j$
- (u, v) and (v, w) are adjacent edges and they lie on a common cycle C of G.

$\Leftarrow$

Proof.

- Assume v is a cut-vertex of G, and let $M = \{c_1, c_2, ..., c_m\}$ be the set of components of $G - v$
- Let $u \in c_i \cap N(v)$ and $w \in c_j \cap N(v)$, where $N(v)$ is the neighbor set of v and $i \neq j$
- (u, v) and (v, w) are adjacent edges and they lie on a common cycle C of G.
- So, $C - \{(u, v), (v, w)\}$ is a path from u to w.

$\Leftarrow$

Proof.

- Assume v is a cut-vertex of G, and let $M = \{c_1, c_2, ..., c_m\}$ be the set of components of $G - v$
- Let $u \in c_i \cap N(v)$ and $w \in c_j \cap N(v)$, where $N(v)$ is the neighbor set of v and $i \neq j$
- (u, v) and (v, w) are adjacent edges and they lie on a common cycle C of G.
- So, $C - \{(u, v), (v, w)\}$ is a path from u to w. **Conflict**!

$\square$

$\Rightarrow$

Proof.

$\Rightarrow$

Proof.

- Assume that there are two adjacent edges (u, v) and (v, w) of G do not lie on any common cycle of G.

$\Rightarrow$

Proof.

- Assume that there are two adjacent edges (u, v) and (v, w) of G do not lie on any common cycle of G.
- Then u and w must be disconnected in $G - v$. Otherwise, there must be one path $P : u \rightsquigarrow w$ in $G - v$, and $P + (u, v) + (v, w)$ must be one cycle of G.

⇒

Proof.

- Assume that there are two adjacent edges (u, v) and (v, w) of G do not lie on any common cycle of G.
- Then u and w must be disconnected in $G - v$. Otherwise, there must be one path $P : u \rightsquigarrow w$ in $G - v$, and $P + (u, v) + (v, w)$ must be one cycle of G. **Conflict!**

□

CZ 5.12

If a connected graph G contains three blocks and k cut-vertices, what are the possible values for k? Explain your answer.

CZ 5.12

If a connected graph G contains three blocks and k cut-vertices, what are the possible values for k? Explain your answer.

- $k = 0$

CZ 5.12

If a connected graph G contains three blocks and k cut-vertices, what are the possible values for k? Explain your answer.

- $k = 0$ ✗

CZ 5.12

If a connected graph G contains three blocks and k cut-vertices, what are the possible values for k? Explain your answer.

- $k = 0$ ✗
- $k = 1$

CZ 5.12

If a connected graph G contains three blocks and k cut-vertices, what are the possible values for k? Explain your answer.

- $k = 0$ ✗
- $k = 1$ ✓

CZ 5.12

If a connected graph G contains three blocks and k cut-vertices, what are the possible values for k? Explain your answer.

- $k = 0$ ✗
- $k = 1$ ✓
- $k = 2$

CZ 5.12

If a connected graph G contains three blocks and k cut-vertices, what are the possible values for k? Explain your answer.

- $k = 0$ ✗
- $k = 1$ ✓
- $k = 2$ ✓

CZ 5.12

If a connected graph G contains three blocks and k cut-vertices, what are the possible values for k? Explain your answer.

- $k = 0$ ✗
- $k = 1$ ✓
- $k = 2$ ✓
- $k \geq 3$

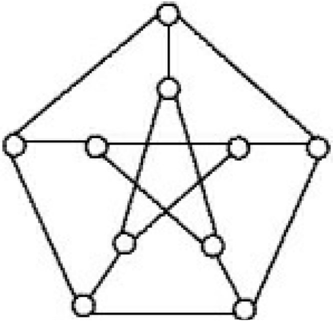
CZ 5.12

If a connected graph G contains three blocks and k cut-vertices, what are the possible values for k? Explain your answer.

- $k = 0$ ✗
- $k = 1$ ✓
- $k = 2$ ✓
- $k \geq 3$ ✗

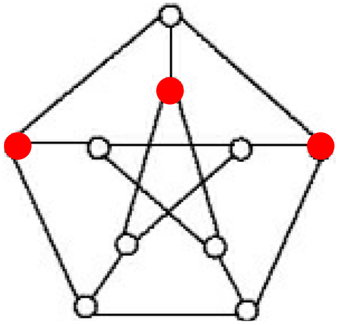
CZ 5.18 (a)

Give an example of a minimum vertex-cut in Petersen graph



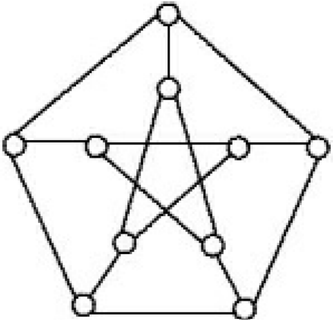
CZ 5.18 (a)

Give an example of a minimum vertex-cut in Petersen graph



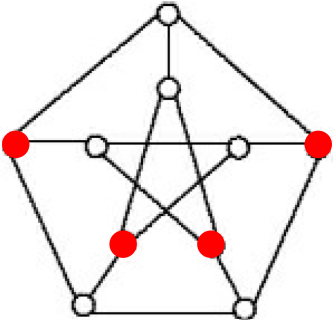
CZ 5.18 (b)

Give an example of vertex-cut U in Petersen graph such that U is not a minimum vertex-cut of Petersen graph and no proper subset of U is a vertex-cut of Petersen graph.



CZ 5.18 (b)

Give an example of vertex-cut U in Petersen graph such that U is not a minimum vertex-cut of Petersen graph and no proper subset of U is a vertex-cut of Petersen graph.



CZ 5.22 (a)

Prove that if G is a k-connected graph and e is an edge of G, then $G - e$ is $(k-1)$-connected.

Proof.

- Assume $G - e$ is not $(k-1)$-connected. Then for any vertex set $W \subset G.V$, s.t. $|W| = k-2$, we have $G - e - W$ is disconnected.
- Let $C = \{C_1, C_2, ..., C_m\}$ be the set of all component of $G - e - W$.

Proof.

- Assume $G - e$ is not $(k-1)$-connected. Then for any vertex set $W \subset G.V$, s.t. $|W| = k - 2$, we have $G - e - W$ is disconnected.
- Let $C = \{C_1, C_2, ..., C_m\}$ be the set of all component of $G - e - W$.
  - $|C| \geq 3$: **impossible!**
  - $|C| = 2$:

Proof.

- Assume $G - e$ is not $(k-1)$-connected. Then for any vertex set $W \subset G.V$, s.t. $|W| = k-2$, we have $G - e - W$ is disconnected.
- Let $C = \{C_1, C_2, ..., C_m\}$ be the set of all component of $G - e - W$.
  - $|C| \geq 3$: **impossible!**
  - $|C| = 2$:
    - $e = (u, v) \Rightarrow u \in C_i, v \in C_j$, where $i \neq j$; otherwise, G cannot be k-connected.

Proof.

- Assume $G - e$ is not $(k-1)$-connected. Then for any vertex set $W \subset G.V$, s.t. $|W| = k-2$, we have $G - e - W$ is disconnected.
- Let $C = \{C_1, C_2, ..., C_m\}$ be the set of all component of $G - e - W$.
  - $|C| \geq 3$: **impossible!**
  - $|C| = 2$:
    - $e = (u, v) \Rightarrow u \in C_i, v \in C_j$, where $i \neq j$; otherwise, G cannot be k-connected.
    - If $|C_i| = |C_j| = 1$, $|G.V| = k$ and $\kappa(G) \leq k-1$. **Conflict!**

Proof.

- Assume $G - e$ is not $(k-1)$-connected. Then for any vertex set $W \subset G.V$, s.t. $|W| = k-2$, we have $G - e - W$ is disconnected.
- Let $C = \{C_1, C_2, ..., C_m\}$ be the set of all component of $G - e - W$.
  - $|C| \geq 3$: **impossible!**
  - $|C| = 2$:
    - $e = (u, v) \Rightarrow u \in C_i, v \in C_j$, where $i \neq j$; otherwise, G cannot be k-connected.
    - If $|C_i| = |C_j| = 1$, $|G.V| = k$ and $\kappa(G) \leq k-1$. **Conflict!**
    - Otherwise, without losing generality, assume $|C_i| \geq 2$. Then, $G - (W + \{u\})$ is not connected and G is not k-connected. **Conflict!**

□

CZ 5.22 (b)

Prove that if G is a k-connected graph and e is an edge of G, then $G - e$ is $(k-1)$-edge-connected.

Proof.

- $$k-1 \leq \kappa(G) \leq \lambda(G) \leq \delta(G)$$

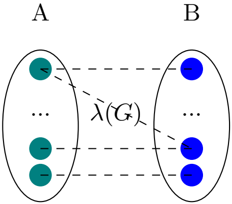
□

CZ 5.26

Prove that if G is a graph of order n such that $\delta(G) \geq (n-1)/2$, then $\lambda(G) = \delta(G)$.

Proof.

- Assume $\lambda(G) < \delta(G)$.
- Let $A \subseteq G.V, B \subseteq G.V$, s.t. $G.V = A \cup B$, $A \cap B = \emptyset$ and $|A| \leq |B|$



$$\begin{aligned}
\lambda(G) &= \sum_{u\in A}\left(\deg(u) - \sum_{v\in A,(u,v)\in G.E} 1\right) \\
&\geq \sum_{u\in A}\left(\delta(G) - \sum_{v\in A,(u,v)\in G.E} 1\right) \\
&\geq \sum_{u\in A}\left(\delta(G) - (|A|-1)\right) \\
&= |A|\delta(G) - |A|(|A|-1)
\end{aligned}$$

$$\delta(G) > \lambda(G) \geq |A|\delta(G) - |A|(|A|-1)$$

$$\delta(G) < |A|$$

- **Case 1:** $|G.V| = n = 2k + 1$

$$\delta(G) < |A| \leq k = (n-1)/2, \text{conflict with } \delta(G) \geq (n-1)/2$$

- **Case 2:** $|G.V| = 2k$

$$\delta(G) < |A| \leq k$$

$$\delta(G) \geq (n-1)/2 \Rightarrow \delta(G) \geq k - 1/2 \Rightarrow \delta(G) \geq k$$

CZ 5.34

Theorem (5.18)

Let G be a k-connected graph and let S be any set of k vertices. If a graph H is obtained from G by adding a new vertex w and joining w to the vertices of S, then H is also k-connected.

Proof.

- Let T be any set of $k-1$ vertices.

- $H-T$ is connected and H is k-connected.

□

Proof.

- Let T be any set of $k-1$ vertices.
  - **Case 1 ($w \notin T$):**
  - **Case 2 ($w \in T$):**
- $H - T$ is connected and H is k-connected.

□

Proof.

- Let T be any set of $k-1$ vertices.
  - **Case 1 ($w \notin T$):**
    - All vertices of G are connected in $H - T$
    - w is connected to at least one vertex of G in $H - T$
    - So, $H - T$ is connected.
  - **Case 2 ($w \in T$):**
- $H - T$ is connected and H is k-connected.

□

Proof.

- Let T be any set of $k-1$ vertices.
  - **Case 1 ($w \notin T$):**
    - All vertices of G are connected in $H-T$
    - w is connected to at least one vertex of G in $H-T$
    - So, $H-T$ is connected.
  - **Case 2 ($w \in T$):**
    - $S' = T - \{w\}$
    - Then, $H - T = G - S'$
    - As $|S'| = k-2$ and G is k-connected, $G - S'$ is connected
    - So, $H-T$ is connected.
- $H-T$ is connected and H is k-connected.

□


Thank
You!