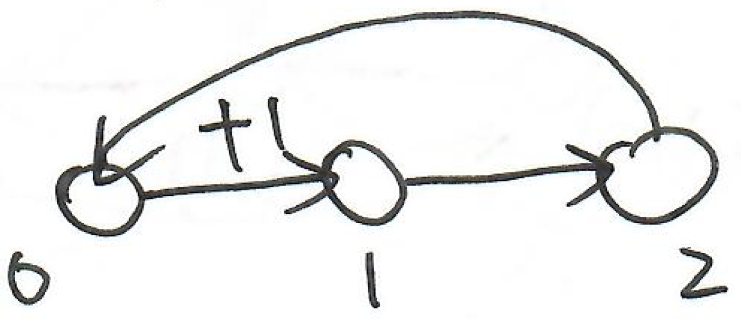
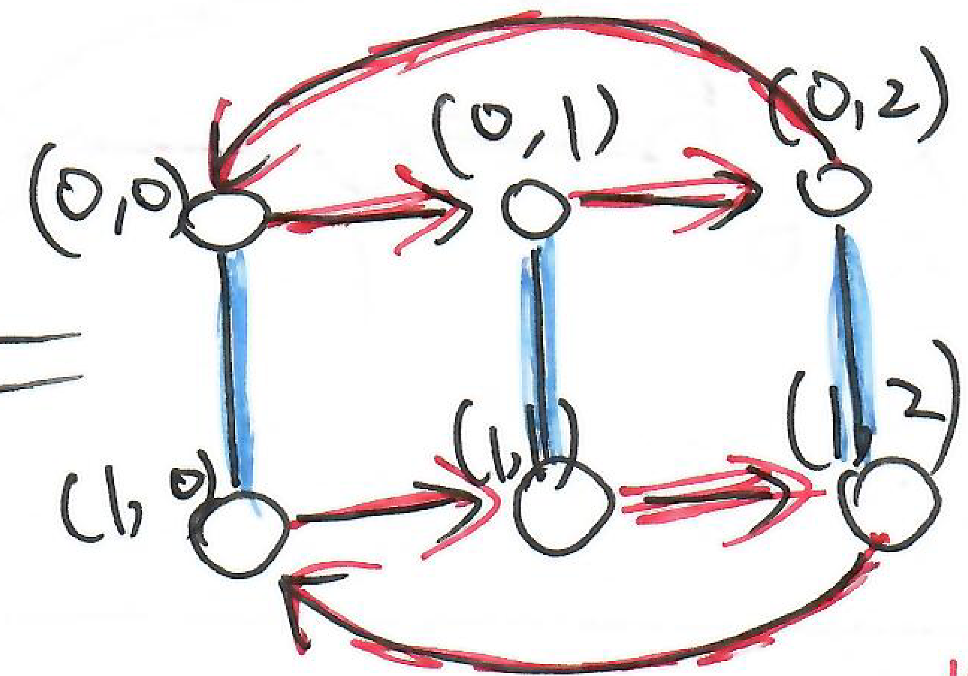
① Direct~~ed~~ Products and Quotients. (2019-04-01)

$\mathbb{Z}_2 \times \mathbb{Z}_3 = G \cong \mathbb{Z}_6$

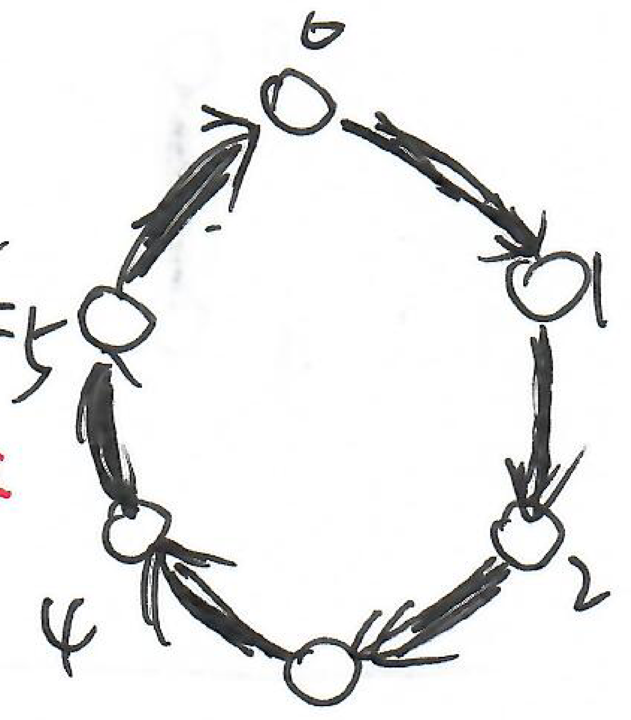
$H \times K$

x

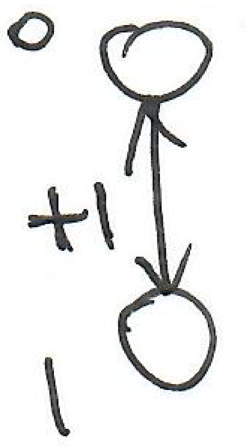




$\cong$ Q



Easy: No restrictions on groups.



$\boxed{Q}$: $G \overset{?}{\cong} \mathbb{Z}_6$ from the point of view of Cayley Diagram?

$(1,1) \to (0,2) \to (1,0) \to (0,1) \to (1,2) \to (0,0)$

$1 \to 2 \to 3 \to 4 \to 5 \to 0$

Another point of view of $\underline{G}$ $(\cong \mathbb{Z}_6)$:

$H \times K = G.$

$H \not\leq G, \ K \not\leq G.$

~~But,~~

Def: $H \times \{e_K\} = H' = \{(h', e_K) : h' \in H\}$ ~~$\leq G$~~

$\{e_H\} \times K = K' = \{(e_H, k') : k' \in K\}$ ~~$\leq G$~~

$\boxed{(1)}$ ~~Furthermore,~~

~~$H' \triangleleft G$.~~

$H' \cong H, \ K' \cong K.$

(2) $H' \leq G, \ K' \leq G$ (Easy!)

$\boxed{(3')}$ $H' \triangleleft G, \ K' \triangleleft G.$ Pf: $(h,k)(h', e_K)(h,k)^{-1} = (hh'h^{-1}, k e_K k^{-1}) \in H \times \{e_K\} = H'.$

(4) $G = H'K'.$

Pf. (1) $H'K' \subseteq G.$

(2) $G \subseteq H'K'.$ $(h,k) = (h, e_K)(e_H, k).$

(5) $H' \cap K' = \{(e_H, e_K)\} = \{e\}.$

(6) H' and K' commute. Pf. $(h', e_K)\cdot(e_H, k') = (h', k') = (e_H \cdot k')\cdot(h' \cdot e_K).$

$\mathbb{Z}_2 \times \mathbb{Z}_3 = G.$

$\mathbb{Z}_2 \not\leq G, \ \mathbb{Z}_3 \not\leq G.$

~~But,~~

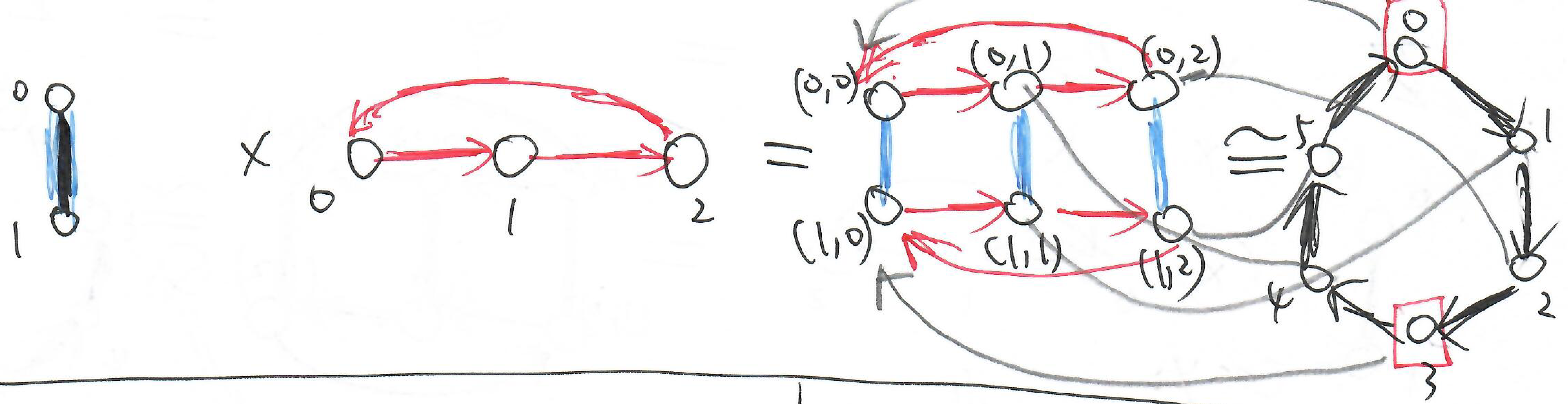
Def. $\mathbb{Z}_2' = \{(0,0), (1,0)\}.$

$\mathbb{Z}_3' = \{(0,0), (0,1), (0,2)\}.$

Thm. If $G = H \times K$, then $\exists H' \cong H, K' \cong K$, $H' \leq G, K' \leq G$ s.t. $G = H'K'$. (G is the internal product of H', K').

Thm. If $G = H \times K$, then $\exists H' \cong H, K' \cong K$, $H' \triangleleft G, K' \triangleleft G$ s.t. $G = H'K'$.

② $\mathbb{Z}_2 \times \mathbb{Z}_3 = G \cong \mathbb{Z}_6$



$G = \mathbb{Z}_2 \times \mathbb{Z}_3$

$G = \mathbb{Z}_2' \mathbb{Z}_3'$

$\mathbb{Z}_6 = \{0,3\}\{0,2,4\}.$

$= \{\underline{0,3},\ \underline{2,5},\ \underline{4,1}\}$

$\underline{\text{Thm 9.27}}.$ $H \triangleleft G,\ K \triangleleft G,$
$G = HK,\quad (H \cap K = \{e\}).$

If G is the internal product of $H \triangleleft G$, and $K \triangleleft G$

$\underline{\text{Then}}$: $G \cong H \times K.$

$\underline{\text{Pf}}.$ $G = HK$

$\forall g \in G.\ \exists h \in H, k \in K : g = hk.$

$G \cong H \times K$

of: $g \mapsto (h,k).$

$D_6 \cong D_3 \times \mathbb{Z}_2.$

$H' \triangleleft G,\quad K' \triangleleft G.$

$H' \cap K' = \{e\}.$

"$\Rightarrow$" $h'k' = k'h'$

$\underbrace{(k')^{-1}\underbrace{h'k'(h')^{-1}}_{\in K}}_{\in H} = e$ $\left.\begin{matrix} \in K \\ \in H \end{matrix}\right\} \Rightarrow \in H \cap K.$

"$\Leftarrow$" $h'k' = k'h'$

$H' \cap K' = \{e\}.$

$g h' g^{-1} = (hk)h'(hk)^{-1}$

$= hkh'k^{-1}h^{-1}$

$= hh'kk^{-1}h^{-1}$

$= hh'h^{-1} \in H$

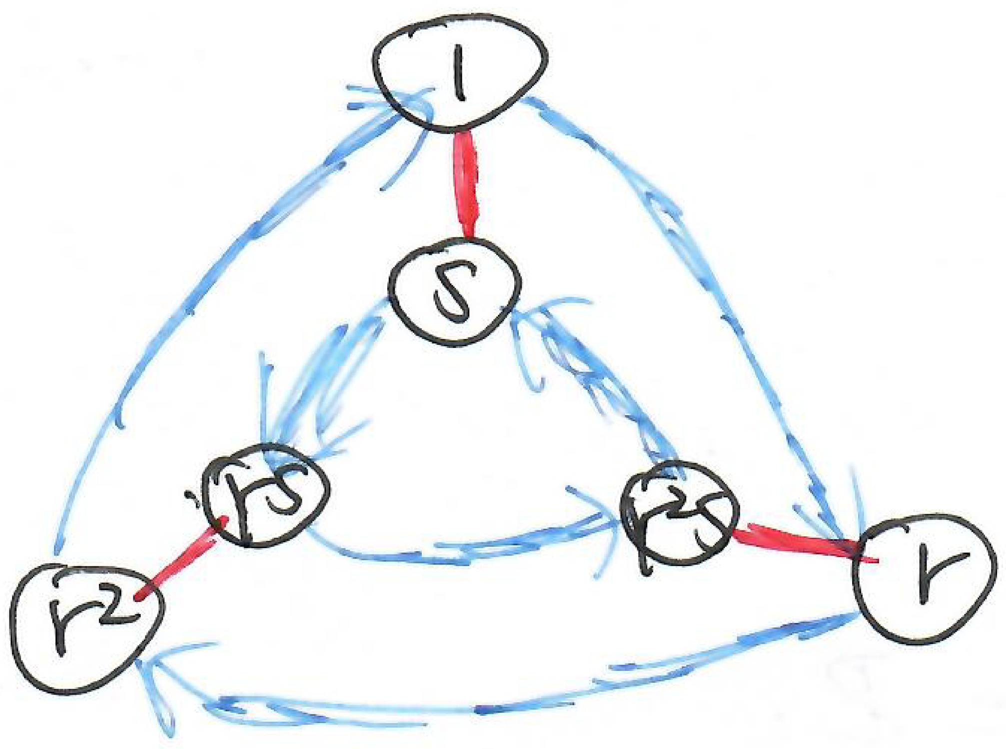
③ $D_6 \cong D_3 \times \mathbb{Z}_2$.

By Thm 9.27:

To find $H \cong D_3, K \cong \mathbb{Z}_2$

$$\begin{cases} H \lhd D_6,\ K \lhd D_6 \\ H \cap K = \{1\}. \\ D_6 = HK. \end{cases}$$

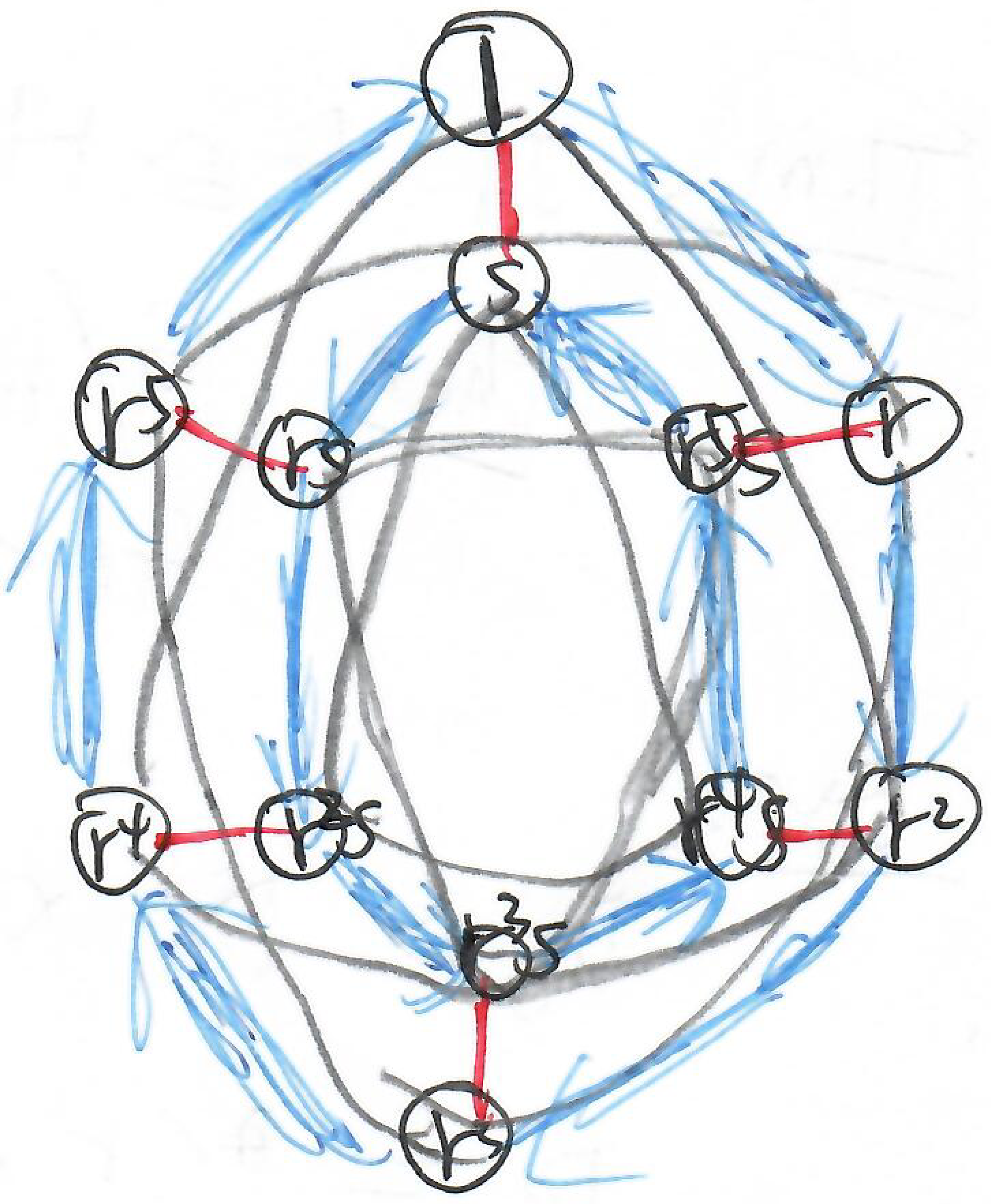
$\underline{D_3}$:



$\times$

$\mathbb{Z}_2$

$\cong$



$\{\boxed{1}, r, \boxed{r^2}, \cancel{r^3}, \boxed{r^4}, r^5, \boxed{s}, rs, \boxed{r^2 s}, r^3 s, \boxed{r^4 s}, r^5 s\}$

$K \cong \mathbb{Z}_2$ $\{1, r^i s\}$.

$K \lhd D_6$ $\{1, r^3\}$.

$= \{1, r^2, r^4, s, r^2 s, r^4 s\}\{1, r^3\}$.

$= \{1, r^2, r^4, s, r^2 s, r^4 s, r^3, r^5, r, r^3 s, r^5 s, rs\}$.

Generally, $D_{2n} \cong D_n \times \mathbb{Z}_2$ (n is odd).

$D_{2n} = \langle r^2, s \rangle \{1, r^n\}$.

$\{1, r^2, r^4, \ldots, s, r^2 s, r^4 s, \ldots\}\{1, r^n\}$.

Another method: $D_{2n} \cong D_n \times \mathbb{Z}_2$

$r^i s^j \mapsto \begin{cases} (r^{i/2} s^j, 0) & i \text{ is even} \\ (r^{i/2} s^j, 1) & i \text{ is odd} \end{cases}$

④

$G = HK$

$$\begin{cases} H \triangleleft G,\ K \triangleleft G, \\ H \cap K = \{1\}. \end{cases}$$

$$\begin{tikzcd}[column sep=small]
& HK \arrow[dl, dash] \arrow[dr, dash] & \\
H \arrow[dr, dash] & & K \arrow[dl, dash] \\
& H\cap K &
\end{tikzcd}$$

The Second Isomorphism Thm:

$$HK/H \cong K/H\cap K \quad \text{i.e., } G/H \cong K$$

$$HK/K \cong H/K\cap K \quad \text{i.e., } G/K \cong H.$$

Thm. If $G \cong H \times K$

then, $G/H\times 1 \cong K$

$G/K\times 1 \cong H$

Ex:

$D_6 \cong D_3 \times \mathbb{Z}_2 \Rightarrow D_6/D_3\times 1 \cong \mathbb{Z}_2$

$D_6 = \langle r^2, s\rangle \{1, r^3\}$.

$$\begin{cases} D_6/\langle r^2, s\rangle \cong \{1, r^3\} \cong \mathbb{Z}_2 \text{ (Explain it in terms of Cayley Diagram)} \\ D_6/\{1, r^3\} \cong \langle r^2, s\rangle \end{cases}$$