

- 教材讨论  
– TC第29章

# 问题1：线性规划的standard和slack form

- 什么是一个linear program？
- 它的standard form有哪些特征？
- 如果一个linear program不具备上述特征，如何将其转化为standard form？

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^n c_j x_j \\ \text{subject to} & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \dots, m \\ & x_j \geq 0 \quad \text{for } j = 1, 2, \dots, n. \end{array}$$

# 问题1: 线性规划的standard和slack form (续)

$$\begin{array}{ll}
 \text{minimize} & -2x_1 + 3x_2 \\
 \text{subject to} & \\
 & x_1 + x_2 = 7 \\
 & x_1 - 2x_2 \leq 4 \\
 & x_1 \geq 0
 \end{array}$$

$$\begin{array}{ll}
 \text{maximize} & 2x_1 - 3x_2 \\
 \text{subject to} & \\
 & x_1 + x_2 = 7 \\
 & x_1 - 2x_2 \leq 4 \\
 & x_1 \geq 0
 \end{array}$$

$$\begin{array}{ll}
 \text{maximize} & 2x_1 - 3x_2' + 3x_2'' \\
 \text{subject to} & \\
 & x_1 + x_2' - x_2'' = 7 \\
 & x_1 - 2x_2' + 2x_2'' \leq 4 \\
 & x_1, x_2', x_2'' \geq 0
 \end{array}$$

$$\begin{array}{ll}
 \text{maximize} & 2x_1 - 3x_2' + 3x_2'' \\
 \text{subject to} & \\
 & x_1 + x_2' - x_2'' \leq 7 \\
 & x_1 + x_2' - x_2'' \geq 7 \\
 & x_1 - 2x_2' + 2x_2'' \leq 4 \\
 & x_1, x_2', x_2'' \geq 0
 \end{array}$$

$$\begin{array}{ll}
 \text{maximize} & 2x_1 - 3x_2 + 3x_3 \\
 \text{subject to} & \\
 & x_1 + x_2 - x_3 \leq 7 \\
 & -x_1 - x_2 + x_3 \leq -7 \\
 & x_1 - 2x_2 + 2x_3 \leq 4 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{array}{ll}
 \text{maximize} & \sum_{j=1}^n c_j x_j \\
 \text{subject to} & \\
 & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \dots, m \\
 & x_j \geq 0 \quad \text{for } j = 1, 2, \dots, n.
 \end{array}$$

# 问题1: 线性规划的standard和slack form (续)

- slack form有哪些特征?  
如何将standard form转化为slack form?

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^n c_j x_j \\ \text{subject to} & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \dots, m \\ & x_j \geq 0 \quad \text{for } j = 1, 2, \dots, n. \end{array}$$



$$\begin{array}{l} z = v + \sum_{j \in N} c_j x_j \\ x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \end{array}$$

in which all variables  $x$  are constrained to be nonnegative.

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} & x_1 + x_2 - x_3 \leq 7 \\ & -x_1 - x_2 + x_3 \leq -7 \\ & x_1 - 2x_2 + 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

# 问题1: 线性规划的standard和slack form (续)

- slack form有哪些特征?  
如何将standard form转化为slack form?

$$\begin{aligned}
 &\text{maximize} && \sum_{j=1}^n c_j x_j \\
 &\text{subject to} && \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \dots, m \\
 &&& x_j \geq 0 \quad \text{for } j = 1, 2, \dots, n.
 \end{aligned}$$



$$\begin{aligned}
 z &= v + \sum_{j \in N} c_j x_j \\
 x_i &= b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B,
 \end{aligned}$$

in which all variables  $x$  are constrained to be nonnegative.

$$\begin{aligned}
 &\text{maximize} && 2x_1 - 3x_2 + 3x_3 \\
 &\text{subject to} && x_1 + x_2 - x_3 \leq 7 \\
 &&& -x_1 - x_2 + x_3 \leq -7 \\
 &&& x_1 - 2x_2 + 2x_3 \leq 4 \\
 &&& x_1, x_2, x_3 \geq 0
 \end{aligned}$$

$$\begin{aligned}
 z &= && 2x_1 - 3x_2 + 3x_3 \\
 x_4 &= && 7 - x_1 - x_2 + x_3 \\
 x_5 &= && -7 + x_1 + x_2 - x_3 \\
 x_6 &= && 4 - x_1 + 2x_2 - 2x_3
 \end{aligned}$$

# 问题2: linear program的应用

- Shortest paths
  - 你能解释为什么这样建模是正确的吗?

$$\begin{array}{ll} \text{maximize} & d_t \\ \text{subject to} & \\ & d_v \leq d_u + w(u, v) \quad \text{for each edge } (u, v) \in E, \\ & d_s = 0. \end{array}$$

## 问题2: linear program的应用 (续)

- 你学会利用linear program来建模实际问题了吗?
  - There are  $m$  different types of food,  $F_1, \dots, F_m$ , that supply varying quantities of the  $n$  nutrients,  $N_1, \dots, N_n$ , that are essential to good health.  
Let  $c_j$  be the minimum daily requirement of nutrient  $N_j$ .  
Let  $b_i$  be the price per unit of food  $F_i$ .  
Let  $a_{ij}$  be the amount of nutrient  $N_j$  contained in one unit of food  $F_i$ .  
The problem is to supply the required nutrients at minimum cost.

## 问题2: linear program的应用 (续)

Let  $y_i$  be the number of units of food  $F_i$  to be purchased per day. The cost per day of such a diet is

$$b_1y_1 + b_2y_2 + \cdots + b_my_m. \quad (1)$$

The amount of nutrient  $N_j$  contained in this diet is

$$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m$$

for  $j = 1, \dots, n$ . We do not consider such a diet unless all the minimum daily requirements are met, that is, unless

$$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \geq c_j \quad \text{for } j = 1, \dots, n. \quad (2)$$

Of course, we cannot purchase a negative amount of food, so we automatically have the constraints

$$y_1 \geq 0, y_2 \geq 0, \dots, y_m \geq 0. \quad (3)$$

Our problem is: minimize (1) subject to (2) and (3). This is exactly the standard minimum problem.



## 问题2: linear program的应用 (续)

- 你学会利用linear program来建模实际问题了吗?
  - There are  $I$  persons available for  $J$  jobs.  
The value of person  $i$  working a whole day at job  $j$  is  $a_{ij}$  for  $i=1, \dots, I$  and  $j=1, \dots, J$ .  
The problem is to choose an assignment of persons to jobs to maximize the total value in one day.  
(Note: A person can work at different jobs at different times of the day.)

## 问题2: linear program的应用 (续)

An assignment is a choice of numbers,  $x_{ij}$ , for  $i = 1, \dots, I$ , and  $j = 1, \dots, J$ , where  $x_{ij}$  represents the proportion of person  $i$ 's time that is to be spent on job  $j$ . Thus,

$$\sum_{j=1}^J x_{ij} \leq 1 \quad \text{for } i = 1, \dots, I \quad (11)$$

$$\sum_{i=1}^I x_{ij} \leq 1 \quad \text{for } j = 1, \dots, J \quad (12)$$

and

$$x_{ij} \geq 0 \quad \text{for } i = 1, \dots, I \text{ and } j = 1, \dots, J. \quad (13)$$

Equation (11) reflects the fact that a person cannot spend more than 100% of his time working, (12) means that only one person is allowed on a job at a time, and (13) says that no one can work a negative amount of time on any job. Subject to (11), (12) and (13), we wish to maximize the total value,

$$\sum_{i=1}^I \sum_{j=1}^J a_{ij} x_{ij}. \quad (14)$$

## 问题2: linear program的应用 (续)

- 你学会利用linear program来建模实际问题了吗?

A company is involved in the production of two items (X and Y). The resources needed to produce X and Y are twofold, namely machine time for automatic processing and craftsman time for hand finishing. The table below gives the number of minutes required for each item:

	Machine time	Craftsman time
X	13	20
Y	19	29

The company has 40 hours of machine time available in the next working week but only 35 hours of craftsman time. Machine time is costed at £10 per hour worked and craftsman time is costed at £2 per hour worked. Both machine and craftsman idle times incur no costs. The revenue received for each item produced (all production is sold) is £20 for X and £30 for Y. The company has a specific contract to produce 10 items of X per week for a particular customer.

Formulate the problem of deciding how much to produce per week as a linear program.

Let  $x$  be the number of items of  $X$ ,  $y$  be the number of items of  $Y$ .

Then the LP is

maximise

$$20x + 30y - 10(\text{machine time worked}) - 2(\text{craftsman time worked})$$

subject to:

$$13x + 19y \leq 40(60) \text{ machine time}$$

$$20x + 29y \leq 35(60) \text{ craftsman time}$$

$$x \geq 10 \text{ contract}$$

$$x, y \geq 0$$

so that the objective function becomes

maximise

$$20x + 30y - \frac{10(13x + 19y)}{60} - \frac{2(20x + 29y)}{60}$$

i.e. maximise

$$17.1667x + 25.8667y$$

subject to:

$$13x + 19y \leq 2400$$

$$20x + 29y \leq 2100$$

$$x \geq 10$$

$$x, y \geq 0$$

## 问题2: linear program的应用 (续)

- 你会利用0-1规划来建模图论问题吗?

$$x \in \{0,1\}$$

- 最大点独立集
- 最大边独立集
- 最小点覆盖集
- 最小边覆盖集
- 旅行商问题

# 问题3: SIMPLEX

- 你能用自己的语言，概述SIMPLEX的思路吗？
  - 关键词1: basic solution
  - 关键词2: pivot

SIMPLEX( $A, b, c$ )

```

1  ( $N, B, A, b, c, v$ ) = INITIALIZE-SIMPLEX( $A, b, c$ )
2  let  $\Delta$  be a new vector of length  $m$ 
3  while some index  $j \in N$  has  $c_j > 0$ 
4      choose an index  $e \in N$  for which  $c_e > 0$ 
5      for each index  $i \in B$ 
6          if  $a_{ie} > 0$ 
7               $\Delta_i = b_i/a_{ie}$ 
8          else  $\Delta_i = \infty$ 
9      choose an index  $l \in B$  that minimizes  $\Delta_l$ 
10     if  $\Delta_l == \infty$ 
11         return "unbounded"
12     else ( $N, B, A, b, c, v$ ) = PIVOT( $N, B, A, b, c, v, l, e$ )
13 for  $i = 1$  to  $n$ 
14     if  $i \in B$ 
15          $\bar{x}_i = b_i$ 
16     else  $\bar{x}_i = 0$ 
17 return ( $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ )
    
```

$$\begin{array}{l}
 \text{maximize} \quad 3x_1 + x_2 + 2x_3 \\
 \text{subject to} \\
 x_1 + x_2 + 3x_3 \leq 30 \\
 2x_1 + 2x_2 + 5x_3 \leq 24 \\
 4x_1 + x_2 + 2x_3 \leq 36 \\
 x_1, x_2, x_3 \geq 0.
 \end{array}$$



$$\begin{array}{l}
 z = 3x_1 + x_2 + 2x_3 \\
 x_4 = 30 - x_1 - x_2 - 3x_3 \\
 x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \\
 x_6 = 36 - 4x_1 - x_2 - 2x_3.
 \end{array}$$



$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}.$$

$$\begin{aligned}
 x_4 &= 30 - x_1 - x_2 - 3x_3 \\
 &= 30 - \left(9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}\right) - x_2 - 3x_3 \\
 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}.
 \end{aligned}$$



$$\begin{aligned}
 z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
 x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
 x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
 x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}.
 \end{aligned}$$

# 问题3: SIMPLEX (续)

- 你理解这步初始化了吗?
  - 如何判定 linear program 是否具有可行解?
  - initial basic solution 如果不是feasible, 怎么办?

INITIALIZE-SIMPLEX ( $A, b, c$ )

```
1 let  $k$  be the index of the minimum  $b_i$ 
2 if  $b_k \geq 0$  // is the initial basic solution feasible?
3 return  $(\{1, 2, \dots, n\}, \{n+1, n+2, \dots, n+m\}, A, b, c, 0)$ 
4 form  $L_{\max}$  by adding  $-x_0$  to the left-hand side of each constraint
  and setting the objective function to  $-x_0$ 
5 let  $(N, B, A, b, c, v)$  be the resulting slack form for  $L_{\max}$ 
6  $l = n + k$ 
7 //  $L_{\max}$  has  $n + 1$  nonbasic variables and  $m$  basic variables.
8  $(N, B, A, b, c, v) = \text{PIVOT}(N, B, A, b, c, v, l, 0)$ 
9 // The basic solution is now feasible for  $L_{\max}$ .
10 iterate the while loop of lines 3–12 of SIMPLEX until an optimal solution
  to  $L_{\max}$  is found
11 if the optimal solution to  $L_{\max}$  sets  $\bar{x}_0$  to 0
12 if  $\bar{x}_0$  is basic
13 perform one (degenerate) pivot to make it nonbasic
14 from the final slack form of  $L_{\max}$ , remove  $x_0$  from the constraints and
  restore the original objective function of  $L$ , but replace each basic
  variable in this objective function by the right-hand side of its
  associated constraint
15 return the modified final slack form
16 else return "infeasible"
```

$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{subject to} & \sum_{j=1}^n a_{ij}x_j - x_0 \leq b_i \quad \text{for } i = 1, 2, \dots, m, \\ & x_j \geq 0 \quad \text{for } j = 0, 1, \dots, n. \end{array}$$

# 问题3: SIMPLEX (续)

- pivot没有变什么? 变化了什么?  
这些不变和变化, 在SIMPLEX中各有什么用?
- SIMPLEX有没有可能永不终止?  
如何判断? 为什么可以这样判断?
- 如何避免这种情况?

$$\begin{aligned} z &= && 3x_1 & + & x_2 & + & 2x_3 \\ x_4 &= 30 & - & x_1 & - & x_2 & - & 3x_3 \\ x_5 &= 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\ x_6 &= 36 & - & 4x_1 & - & x_2 & - & 2x_3 \end{aligned}$$



$$\begin{aligned} z &= 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\ x_1 &= 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\ x_4 &= 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\ x_5 &= 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2} \end{aligned}$$