作业1-11

UD第20章问题4、8、**9**、10

UD第21章问题7、9、**10**、**11**、**16**、17、18、19

UD第22章问题**1**、2、**3**、6、9

UD第23章问题2、3、10

Problem 20.9. (a) Suppose that A and B are nonempty finite sets and $A \cap B = \emptyset$. Show that there exist positive integers n and m such that $A \approx \{1, 2, \ldots, n\}$ and $B \approx \{n+1, \ldots, n+m\}$.

(b) Prove Corollary **20.4**

Let A and B be disjoint sets. If A and B are finite, then $A \cup B$ is finite.

(a)

$\because$ **A, B are nonempty finite sets**

$\therefore \exists n, m \in \mathbb{Z}^+, s.t.,$

$$A \approx \{1,2, \ldots, n\}$$

$$B \approx \{1,2, \ldots, m\}, \text{in another word}, \exists g: B \to \{1,2, \ldots, m\} \text{ and } g \text{ is bijective}$$

The function $f(x) = x + n$ from$\{1,2, \ldots m\}$ to $\{\text{n} + 1, \ldots, \text{n} + \text{m}\}$, we could show that $f(x)$ is bijective (skipped).

$\because g, f$ are both bijective

$\therefore f \circ g: B \to \{n+1, \ldots, n+m\}$ is bijective, then $B \approx \{n+1, \ldots, n+m\}$

(b)

Case1: if A or B is empty, $A \cup B$ is obviously finite;

Case2: neither A nor B is empty

$\because A, B$ are nonempty finite sets

$\therefore$ from (a), we know that $\exists n, m \in \mathbb{Z}^+, s.t.,$

$$A \approx \{1,2, \ldots, n\}$$

$$B \approx \{n+1, \ldots, n+m\}$$

$\because A \cap B = \emptyset$ and $\{1,2, \ldots, \text{n}\} \cap \{n+1, \ldots, n+m\}$, By Theorem 20.6, we have:

$$A \cup B \approx \{1,2, \ldots, n\} \cup \{n+1, \ldots, n+m\} = \{1,2, \ldots, n+m\}$$

$\therefore A \cup B$ is finite

Theorem 20.6 *Let A, B, C, and D be nonempty sets. If $A \cap B = \emptyset$, $C \cap D = \emptyset$, $A \approx C$, and $B \approx D$, then $A \cup B \approx C \cup D$.*

Problem 21.10. Suppose that A is an infinite set, B is a finite set, and $f : A \to B$ is a function. Show that there exists $b \in B$ such that $f^{-1}(\{b\})$ is infinite.

$\because f$ is well defined
$\therefore \bigcup_{b \in B} f^{-1}(\{b\}) = A$

Assume that $\forall b \in B, f^{-1}(\{b\})$ is finite.
According to Exercise 20.13, we could conclude that $\bigcup_{b \in B} f^{-1}(\{b\})$ is finite(Contradiction)
So, the assumption is not correct, $\exists b \in B$ $s.t.$ $f^{-1}(\{b\})$ is finite

Exercise 20.13. Use induction to prove the following. Let $m \in \mathbb{Z}^+$. If $A_1, A_2, \ldots, A_m$ are finite sets, then the union $\bigcup_{j=1}^{m} A_j$ is finite.

Can be proved by Introduction on m

Problem 21.11. Let X be an infinite set, and A and B be finite subsets of X. Answer each of the following, giving reasons for your answers:

(e) If $f : A \to X$ is a one-to-one function, is $f(A)$ finite or infinite?

Obviously, $ran(f) = f(A) \subseteq X$
Define a function, $g: A \to f(A)$ as $g(x) = f(x)$
It is easy to show that g is one-to-one and onto, so g is bijective
Then, $A \approx f(A)$
$f(A)$ is finite

Problem 21.15. Let A be a nonempty finite set with $|A| = n$ and let $a \in A$. Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

$\because$ A is nonempty and finite with $|A| = n$
$\therefore \exists f: A \rightarrow \{1,2,\ldots,n\}$ and f is bijective
For each $a \in A$, $f(a) \in \{1,2,\ldots,n\}$, we can define an new function $g: A\backslash\{a\} \rightarrow \{1,2,\ldots,n\}\backslash\{f(a)\}$ as following:

$$g(x) = f(x), x \in A\backslash\{a\}$$

We can show that $g(x)$ is bijective(skipped), so $A\backslash\{a\} \approx \{1,2,\ldots,n\}\backslash f(a)$
Given $f(a)$ we could also find a function $h: \{1,2,\ldots,n\}\backslash f(a) \rightarrow \{1,2,\ldots,\mathrm{n}-1\}$ as:

$$h(x) = \begin{cases} f(x), & x < f(a) \\ f(x) - 1, & x > f(a) \end{cases}$$

We can show that $h(x)$is bijective, so $\{1,2,..,\mathrm{n}\}\backslash f(a) \approx \{1,2,\ldots,n-1\}$
Consequently, $A\backslash\{a\} \approx \{1,2,\ldots,n-1\}$, so $|A\backslash\{a\}| = n - 1$

Problem 21.16. (a) Suppose that A is a finite set and $B \subseteq A$. We showed that B is finite. Show that $|B| \leq |A|$.

(b) Suppose that A is a finite set and $B \subseteq A$. Show that if $B \neq A$, then $|B| < |A|$.

(c) Show that if two finite sets A and B satisfy $B \subseteq A$ and $|A| \leq |B|$, then $A = B$.

(a)

Case1: $B = \emptyset$, obviously $|B| \leq |A|$

Case2: $B \neq \emptyset$, obviously A is not empty.

Case 2.1: $A = B$, obviously $|B| = |A|$

Case 2.2: $B \subset A$,then$|B| = |A| - |A \backslash \mathrm{B}| < |A|$

we now try to show: For an arbitrary non empty finite set B and any of its finite superset A, then $|A \backslash \mathrm{B}| = |A| - |B|$

by introduction on |B|

base case: $|\mathrm{B}| = 1$, by problem 21.15, we get $|A \backslash \mathrm{B}| = |A| - 1$

H:for each $|B| \leq k$, $|A \backslash \mathrm{B}| = |A| - |B|$

I: when$|B| = k + 1$, let $x \in B$ be an arbitrary element of B, then $B = (B \backslash \{x\}) \cup \{x\}$

so $A \backslash \mathrm{B} = A \backslash ((B \backslash \{x\}) \cup \{x\}) = (A \backslash \{x\}) \backslash (B \backslash \{x\})$

From base case, we know that $|A \backslash \{x\}| = |A| - 1$, $|B \backslash \{x\}| = |B| - 1$

As $|B \backslash \{x\}| = |B| - 1$, and $\mathrm{B} \backslash \{x\} \subseteq A \backslash \{x\}$, from H, we know that

$|(A \backslash \{x\}) - B \backslash \{x\}| = |A \backslash \{x\}| - |B \backslash \{x\}| = |A| - |B|$;

Problem 20.9. (a) Suppose that A and B are nonempty finite sets and $A \cap B = \emptyset$. Show that there exist positive integers n and m such that $A \approx \{1, 2, \ldots, n\}$ and $B \approx \{n+1, \ldots, n+m\}$.

(b) Prove Corollary **20.4**

Let A and B be disjoint sets. If A and B are finite, then $A \cup B$ is finite.

(a)
$\because$ A, B are nonempty finite sets
$\therefore \exists n, m \in \mathbb{Z}^+, s.t.,$

$$A \approx \{1,2, \ldots, n\}$$
$$B \approx \{1,2, \ldots, m\}$$, in another word, $\exists g: B \to \{1,2, \ldots, m\}$ and g is bijective

The function $f(x) = x + n$ from$\{1,2, \ldots m\}$ to $\{n+1, \ldots, n+m\}$, we could show that $f(x)$ is bijective(skipped).
$\because$ g, f are both bijective
$\therefore f \circ g: B \to \{n+1, \ldots, n+m\}$ is bijective, then $B \approx \{n+1, \ldots, n+m\}$

(b)
Case1: if A or B is empty, $A \cup B$ is obviously finite;
Case2: neither A nor B is empty
$\because$ A, B are nonempty finite sets
$\therefore$ from (a), we know that $\exists n, m \in \mathbb{Z}^+, s.t.,$

$$A \approx \{1,2, \ldots, n\}$$
$$B \approx \{n+1, \ldots, n+m\}$$

$\because$ $A \cap B = \emptyset$ and $\{1,2, \ldots, n\} \cap \{n+1, \ldots, n+m\}$, By Theorem 20.6, we have:

$$A \cup B \approx \{1,2, \ldots, n\} \cup \{n+1, \ldots, n+m\} = \{1,2, \ldots, n+m\}$$

$\therefore$ $A \cup B$ is finite

$$\boldsymbol{A = A \backslash B \cup B, |A \backslash B| \geq 0}$$

回顾

Theorem 20.6 *Let A, B, C, and D be nonempty sets. If $A \cap B = \emptyset$, $C \cap D = \emptyset$, $A \approx C$, and $B \approx D$, then $A \cup B \approx C \cup D$.*

Problem 21.16. (a) Suppose that A is a finite set and $B \subseteq A$. We showed that B is finite. Show that $|B| \leq |A|$.
(b) Suppose that A is a finite set and $B \subseteq A$. Show that if $B \neq A$, then $|B| < |A|$.
(c) Show that if two finite sets A and B satisfy $B \subseteq A$ and $|A| \leq |B|$, then $A = B$.

(a)
Case1: $B = \emptyset$, obviously $|B| \leq |A|$
Case2: $B \neq \emptyset$, obviously A is not empty.

There is a *bijective* function $g: A \to \{1,2,..,|A|\}$, and a *bijective* function $h: B \to \{1,2,..,|B|\}$
Then the function H=$g \circ f \circ h^{-1}$ should be 1-to-1

$\because B \subseteq A$
We could find a function $f: B \to A$ as $f(x) = x$
Obviously, f is 1-to-1
Assume that $|B| > |A|$, then according to **Pigeonhole principle** **H** cannot be 1-to-1, which is contractive to our assumption.
Consequently, $|\boldsymbol{B}| \leq |\boldsymbol{A}|$

Theorem 22.2 (Pigeonhole principle). *Let m and n be positive integers with $m > n$, and let f be a map satisfying $f: \{1,\ldots,m\} \to \{1,\ldots,n\}$. Then f is not one-to-one.*

Problem 22.1 Give an example, if possible, of each of the following:

(a) a countably infinite collection of pairwise disjoint finite sets whose union is countably infinite (see Problem **8.11** for the definition of pairwise disjoint);
(b) a countably infinite collection of nonempty sets whose union is finite;
(c) a countably infinite collection of pairwise disjoint nonempty sets whose union is finite.

For(a), For $n \in \mathbb{N}$ let $A_n = \{n\}$. Then $A_n \cap A_m = \emptyset$ for $n \neq m$ as required, and $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{N}$

For (b), try something like $A_n = \{1\}$. Then $\bigcup A_n = \{1\}$ is finite.

For (c), no example exists: If I is an index set and for each $i \in I$, A_i is a nonempty set, then by picking an element $a_i \in A_i$, we obtain a map $I \to \bigcup_{i\in I} A_i$. If the A_i are pairwise disjoint, this map is certainly injective, hence $|\bigcup_{i\in I} A_i| \geq |I|$

Problem 22.3 . Is the set of all infinite sequences of 0's and 1's finite, countably infinite, or uncountable? Guess and then prove, please.

Uncountable!

Let A be the set of all infinite sequences of 0's and 1's, then A is obviously **infinite**

Assume A is countable, then $A \approx \mathbb{N}$, then there should be a bijective function $f: \mathbb{N} \to A$

$$f(1) = \boldsymbol{a_{11}} a_{12} a_{13} \ldots$$
$$f(2) = a_{21} \boldsymbol{a_{22}} a_{23} \ldots$$
$$f(3) = a_{31} a_{32} \boldsymbol{a_{33}} \ldots$$
$$\ldots$$

Here, $a_{ij} \in \{0,1\}$.

Then we could construct an infinite sequence of 0's and 1's ($x = b_1 b_2 b_3 \ldots$)by:

$$b_i = 1 - a_{ii}, i = 1,2,3 \ldots$$

Obviously $x \in A$, but $\forall a \in A, f(a) \neq x$

Therefore, f cannot be onto, so cannot be bijective.

Problem 22.3. Is the set of all infinite sequences of 0's and 1's finite, countably infinite, or uncountable? Guess and then prove, please.

which finally ends with infinite 0's

Let A be the set of all infinite sequences of 0's and 1's which finally ends with infinite 0's, then A is obviously **infinite**

For an arbitrary $x \in A$, x finally ends with infinite 0's, let n_x be the position of the last 1 in x. Then x can be represented as:

$$x = a_1 a_2 \dots a_{n_x} 000 \dots$$

So, for an arbitrary $x \in A$, we could mapping it to an finite sequence of 0's and 1's by function $f: A \to B$, *where B is the set of all* finite sequences of 0's and 1's that **start with 0:**

$$f(x) = \mathbf{0} a_{n_x} a_{n_x - 1} \dots a_2 a_1, \text{where } x = a_1 a_2 \dots a_{n_x} 000 \dots$$

For each element $b \in B$, b can be seen as the binary representation of an Natural Number, and $B \approx \mathbb{N}$

It is easy to show that f is one-to-one(skipped), so **A is countable！** （**By Exercise 22.5**）

Exercise 22.5 Prove that a nonempty set A is countable if and only if there exists a one-to-one function $f: A \to \mathbb{N}$. ○

$\aleph_0$与$\aleph_1$之间还有什么？

- 在数学中，**连续统假设**（英语：Continuum hypothesis，简称CH）是一个猜想，也是希尔伯特的23个问题的第一题，由康托尔提出，关于无穷集的可能大小。其为：
  - **不存在一个基数绝对大于可列集($\aleph_0$)而绝对小于实数集($\aleph_1$)的集合。**

讨论，$\mathbb{N}, P(\mathbb{N}), \mathbb{R}$与$\aleph_0, \aleph_1$

已知$[0,1] \approx \mathbb{R}$, 所以仅需考虑$[0,1]$
找到一个双射 function $f: P(\mathbb{N}) \rightarrow [0,1]$

- $|P(\mathbb{N})| \ ? \ |\mathbb{R}|$

能找到么？

找到一个双射 function $f: P(\mathbb{N}) \to [0,1]$

- 基本思路：
  - 令A为所有仅由0,1所构成的序列所构成的集合。
  - 对于任意元素$X \in P(\mathbb{N}), X \subseteq \mathbb{N}$,构造$g: P(\mathbb{N}) \to A$
    - $g(X) = a_1 a_2, \ldots, a_n \ldots$,其中$a_i = 1 \text{ iff } i \in X, \text{else } a_i = 0$
  - 易证：g为双射 在此处键入公式。
  - 下证$A \approx [0,1]$, 定义函数$h: A \to [0,1]$：

$$h(a) = \sum_{i \in \mathbb{N}} a_i \left(\frac{1}{2}\right)^i$$

易证h同为双射

$\mathbf{0.}\, \boldsymbol{a_1 a_2, \ldots, a_n \ldots}$ **[0,1]区间实数二进制表示**