- 教材讨论
  - TC第29章

问题1：线性规划的standard和slack form

- 什么是一个linear program？
- 它的standard form有哪些特征？
- 如果一个linear program不具备上述特征，如何将其转化为standard form？

$$
\begin{aligned}
&\text{maximize} \quad \sum_{j=1}^{n} c_j x_j \\
&\text{subject to} \\
&\qquad \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for } i = 1, 2, \ldots, m \\
&\qquad\qquad x_j \ge 0 \quad \text{for } j = 1, 2, \ldots, n .
\end{aligned}
$$

问题1：线性规划的standard和slack form (续)

$$\begin{array}{lrcrcl} \text{minimize} & -2x_1 & + & 3x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array}$$

$$\begin{array}{lrcrcl} \text{maximize} & 2x_1 & - & 3x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array}$$

$$\begin{array}{lrcrcrcl} \text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & x_2' & - & x_2'' & = & 7 \\ & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\ & & & x_1, x_2', x_2'' & & & \geq & 0 \end{array}$$

$$\begin{array}{lrcrcrcl} \text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\ & x_1 & + & x_2' & - & x_2'' & \geq & 7 \\ & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\ & & & x_1, x_2', x_2'' & & & \geq & 0 \end{array}$$

$$\begin{array}{lrcrcrcr} \text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & x_2 & - & x_3 & \leq & 7 \\ & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\ & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\ & & & x_1, x_2, x_3 & & & \geq & 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} & \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \ldots, m \\ & x_j \geq 0 \quad \text{for } j = 1, 2, \ldots, n\,. \end{array}$$

问题1：线性规划的standard和slack form (续)

- slack form有哪些特征？
  如何将standard form转化为slack form？

$$\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \\
& \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for } i = 1, 2, \ldots, m \\
& x_j \ge 0 \quad \text{for } j = 1, 2, \ldots, n .
\end{aligned}$$

→

$$\begin{aligned}
z &= v + \sum_{j \in N} c_j x_j \\
x_i &= b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B ,
\end{aligned}$$

in which all variables x are constrained to be nonnegative.

$$\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \le & 7 \\
& -x_1 & - & x_2 & + & x_3 & \le & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \le & 4 \\
& & & x_1, x_2, x_3 & & & \ge & 0
\end{array}$$

问题1：线性规划的standard和slack form (续)

- slack form有哪些特征？
  如何将standard form转化为slack form？

$$\begin{aligned}\text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \\ & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for } i = 1, 2, \ldots, m \\ & x_j \ge 0 \quad \text{for } j = 1, 2, \ldots, n .\end{aligned}$$

$$\begin{aligned} z &= v + \sum_{j \in N} c_j x_j \\ x_i &= b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B , \end{aligned}$$

in which all variables x are constrained to be nonnegative.

$$\begin{aligned}\text{maximize} \quad & 2x_1 - 3x_2 + 3x_3 \\ \text{subject to} \quad & \\ & x_1 + x_2 - x_3 \le 7 \\ & -x_1 - x_2 + x_3 \le -7 \\ & x_1 - 2x_2 + 2x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

$$\begin{aligned} z &= 2x_1 - 3x_2 + 3x_3 \\ x_4 &= 7 - x_1 - x_2 + x_3 \\ x_5 &= -7 + x_1 + x_2 - x_3 \\ x_6 &= 4 - x_1 + 2x_2 - 2x_3 \end{aligned}$$

问题2：linear program的应用

- Shortest paths
  - 你能解释为什么这样建模是正确的吗？

$$
\begin{aligned}
\text{maximize} \quad & d_t \\
\text{subject to} \quad & \\
& d_v \le d_u + w(u,v) \quad \text{for each edge } (u,v) \in E \,, \\
& d_s = 0 \,.
\end{aligned}
$$

问题2：linear program的应用 (续)

- 你学会利用linear program来建模实际问题了吗？
  - There are m different types of food, $F_1,...,F_m$, that supply varying quantities of the n nutrients, $N_1,...,N_n$, that are essential to good health.
    Let c_j be the minimum daily requirement of nutrient N_j .
    Let b_i be the price per unit of food F_i.
    Let a_{ij} be the amount of nutrient N_j contained in one unit of food F_i.
    The problem is to supply the required nutrients at minimum cost.

问题2：linear program的应用 (续)

Let y_i be the number of units of food F_i to be purchased per day. The cost per day of such a diet is

$$b_1 y_1 + b_2 y_2 + \cdots + b_m y_m. \tag{1}$$

The amount of nutrient N_j contained in this diet is

$$a_{1j} y_1 + a_{2j} y_2 + \cdots + a_{mj} y_m$$

for $j = 1, \ldots, n$. We do not consider such a diet unless all the minimum daily requirements are met, that is, unless

$$a_{1j} y_1 + a_{2j} y_2 + \cdots + a_{mj} y_m \geq c_j \qquad \text{for } j = 1, \ldots, n. \tag{2}$$

Of course, we cannot purchase a negative amount of food, so we automatically have the constraints

$$y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0. \tag{3}$$

Our problem is: minimize (1) subject to (2) and (3). This is exactly the standard minimum problem.

问题2：linear program的应用 (续)

- 你学会利用linear program来建模实际问题了吗？
  - There are I persons available for J jobs.
    The value of person i working a whole day at job j is a_{ij} for i=1,...,I and j=1,...,J.
    The problem is to choose an assignment of persons to jobs to maximize the total value in one day.
    (Note: A person can work at different jobs at different times of the day.)

问题2：linear program的应用 (续)

An assignment is a choice of numbers, x_{ij}, for $i = 1, \ldots, I$, and $j = 1, \ldots, J$, where x_{ij} represents the proportion of person i's time that is to be spent on job j. Thus,

$$\sum_{j=1}^{J} x_{ij} \leq 1 \qquad \text{for } i = 1, \ldots, I \tag{11}$$

$$\sum_{i=1}^{I} x_{ij} \leq 1 \qquad \text{for } j = 1, \ldots, J \tag{12}$$

and

$$x_{ij} \geq 0 \qquad \text{for } i = 1, \ldots, I \text{ and } j = 1, \ldots, J. \tag{13}$$

Equation (11) reflects the fact that a person cannot spend more than 100% of his time working, (12) means that only one person is allowed on a job at a time, and (13) says that no one can work a negative amount of time on any job. Subject to (11), (12) and (13), we wish to maximize the total value,

$$\sum_{i=1}^{I} \sum_{j=1}^{J} a_{ij} x_{ij}. \tag{14}$$

问题2：linear program的应用 (续)

- 你学会利用linear program来建模实际问题了吗？

A company is involved in the production of two items (X and Y). The resources need to produce X and Y are twofold, namely machine time for automatic processing and craftsman time for hand finishing. The table below gives the number of minutes required for each item:

	Machine time	Craftsman time
X	13	20
Y	19	29

The company has 40 hours of machine time available in the next working week but only 35 hours of craftsman time. Machine time is costed at £10 per hour worked and craftsman time is costed at £2 per hour worked. Both machine and craftsman idle times incur no costs. The revenue received for each item produced (all production is sold) is £20 for X and £30 for Y. The company has a specific contract to produce 10 items of X per week for a particular customer.

Formulate the problem of deciding how much to produce per week as a linear program.

Let x be the number of items of X, y be the number of items of Y.

Then the LP is

maximise

$$20x + 30y - 10(machine\ time\ worked) - 2(craftsman\ time\ worked)$$

subject to:

$$13x + 19y \leq 40(60)\ machine\ time$$

$$20x + 29y \leq 35(60)\ craftsman\ time$$

$$x \geq 10\ contract$$

$$x, y \geq 0$$

so that the objective function becomes

maximise

$$20x + 30y - \frac{10(13x + 19y)}{60} - \frac{2(20x + 29y)}{60}$$

i.e. maximise

$$17.1667x + 25.8667y$$

subject to:

$$13x + 19y \leq 2400$$

$$20x + 29y \leq 2100$$

$$x \geq 10$$

$$x, y \geq 0$$

问题3：SIMPLEX

- 你能用自己的语言，概述SIMPLEX的思路吗？
  - 关键词1：basic solution
  - 关键词2：pivot

```
SIMPLEX(A, b, c)
(N, B, A, b, c, v) = INITIALIZE-SIMPLEX(A, b, c)
let Δ be a new vector of length m
while some index j ∈ N has c_j > 0
    choose an index e ∈ N for which c_e > 0
    for each index i ∈ B
        if a_ie > 0
            Δ_i = b_i / a_ie
        else Δ_i = ∞
    choose an index l ∈ B that minimizes Δ_l
    if Δ_l == ∞
        return "unbounded"
    else (N, B, A, b, c, v) = PIVOT(N, B, A, b, c, v, l, e)
for i = 1 to n
    if i ∈ B
        x̄_i = b_i
    else x̄_i = 0
return (x̄_1, x̄_2, ..., x̄_n)
```

$$\begin{array}{lrcrcrcl} \text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & x_2 & + & 3x_3 & \le & 30 \\ & 2x_1 & + & 2x_2 & + & 5x_3 & \le & 24 \\ & 4x_1 & + & x_2 & + & 2x_3 & \le & 36 \\ & & & x_1, x_2, x_3 & & & \ge & 0 . \end{array}$$

$$\begin{array}{rcrcrcrcr} z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\ x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\ x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 . \end{array}$$

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} .$$

$$\begin{aligned} x_4 &= 30 - x_1 - x_2 - 3x_3 \\ &= 30 - \left(9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}\right) - x_2 - 3x_3 \\ &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} . \end{aligned}$$

$$\begin{array}{rcrcrcrcr} z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\ x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\ x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\ x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2} . \end{array}$$

问题3：SIMPLEX (续)

- 你理解这步初始化了吗？
  - 如何判定 linear program 是否具有可行解？
  - initial basic solution 如果不是feasible，怎么办？

INITIALIZE-SIMPLEX(A, b, c)

let k be the index of the minimum b_i
if $b_k \geq 0$ // is the initial basic solution feasible?
    return $(\{1, 2, \ldots, n\}, \{n+1, n+2, \ldots, n+m\}, A, b, c, 0)$
form L_{aux} by adding $-x_0$ to the left-hand side of each constraint and setting the objective function to $-x_0$
let (N, B, A, b, c, v) be the resulting slack form for L_{aux}
$l = n + k$
// L_{aux} has $n+1$ nonbasic variables and m basic variables.
$(N, B, A, b, c, v) = \text{PIVOT}(N, B, A, b, c, v, l, 0)$
// The basic solution is now feasible for L_{aux}.
iterate the **while** loop of lines 3–12 of SIMPLEX until an optimal solution to L_{aux} is found
if the optimal solution to L_{aux} sets $\bar{x}_0$ to 0
    if $\bar{x}_0$ is basic
        perform one (degenerate) pivot to make it nonbasic
    from the final slack form of L_{aux}, remove x_0 from the constraints and restore the original objective function of L, but replace each basic variable in this objective function by the right-hand side of its associated constraint
    return the modified final slack form
else return "infeasible"

$$
\begin{aligned}
&\text{maximize} && -x_0 \\
&\text{subject to} && \\
&&& \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i \quad \text{for } i = 1, 2, \ldots, m, \\
&&& x_j \geq 0 \quad \text{for } j = 0, 1, \ldots, n.
\end{aligned}
$$

问题3：SIMPLEX (续)

- pivot没有变什么？变化了什么？
  这些不变和变化，在SIMPLEX中各有什么用？
- pivot有没有可能永不终止？
  如何判断？为什么可以这样判断？
- 如何避免这种情况？

$$
\begin{aligned}
z &= 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3 .
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} .
\end{aligned}
$$