2-8 Probabilistic Analysis

"No Expectation, No Disappointment."

Hengfeng Wei

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April 21, 2020

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RANDOMIZE-IN-PLACE(A)
n = A.length
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    swap A[i] with A[RANDOM(i, n)]
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Sampling without Replacement

Permute-With-All (TC 5.3-3)

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N^N *vs.* $N!$

$3^3 = 27$ *vs.* $3! = 6$

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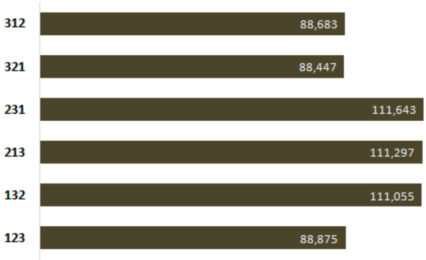

$\# = 600,000$

The Danger of Naïveté @ Coding Horror

Definition (Expectation)

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Theorem (A Second Way of Computing Expectation)

$$\mathbb{E}[X] = \sum_{s \in S} X(s) \Pr(s)$$

Theorem (Linearity of Expectation)

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

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$$\mathbb{E}[X + Y] = \sum_{s \in S} \Big(X(s) + Y(s) \Big) \Pr(s) = \mathbb{E}[X] + \mathbb{E}[Y]$$

Definition (Indicator Random Variable)

$$I_E = \begin{cases} 1, & \text{if } E \text{ occurs} \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbb{E}[I_E] = \Pr(E)$$

Hat-check Problem (TC 5.2-4)

X : # of customers who get back their own hat $\mathbb{E}[X]$

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$$\mathbb{E}[X] = 1$$

Inversions (TC 5.2-5)

$A[1 \cdots n]$ of n distinct numbers

(i, j) is an inversion of $A : i < j \wedge A[i] > A[j]$

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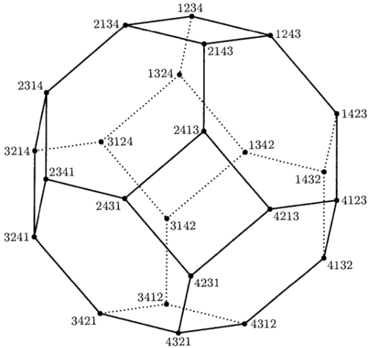

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\begin{aligned}
\mathbb{E}[X] &= \sum_{s \in S} X(s)\Pr(s) = \frac{1}{n!}\sum_{s \in S} X(s) \\
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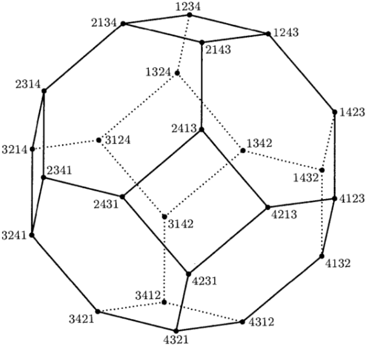

# of inversions in $\langle 3214 \rangle$ + # of inversions in $\langle 4123 \rangle$

Searching an Unsorted Array (TC Problem 5-2 (f))

```
procedure DETERMINISTIC-SEARCH(A[1 ··· n], x)
    i ← 1
    while i ≤ n do
        if A[i] = x then
            return true
        i ← i + 1
    return false
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\end{aligned}$$

$$k = 1 \implies \mathbb{E}[Y] = \frac{n+1}{2}, \qquad k = n \implies \mathbb{E}[Y] = 1$$

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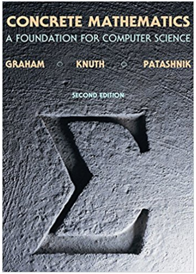

$$r\binom{r-1}{k-1} = k\binom{r}{k}$$

$$\sum_{0\le k\le n}\binom{k}{m} = \binom{n+1}{m+1}$$

```
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```

$$
\begin{aligned}
\sum_{i=1}^{n-k+1} i\binom{n-i}{k-1} &= \sum_{i=0}^{n-k}(i+1)\binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k}\left((n+1)-(n-i)\right)\binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k}(n+1)\binom{n-i-1}{k-1} - \sum_{i=0}^{n-k}(n-i)\binom{n-i-1}{k-1} \\
&= (n+1)\sum_{i=0}^{n-k}\binom{n-i-1}{k-1} - k\sum_{i=0}^{n-k}\binom{n-i}{k} \\
&= (n+1)\sum_{m=k-1}^{n-1}\binom{m}{k-1} - k\sum_{m=k}^{n}\binom{m}{k} \\
&= (n+1)\binom{n}{k} - k\binom{n+1}{k+1} = \binom{n+1}{k+1}
\end{aligned}
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Theorem (A Third Way of Computing Expectation)

Let X be a discrete random variable that takes on only nonnegative integer values $\mathbb{N}$.

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} i \Pr(X = i) = \sum_{i=1}^{\infty} \Pr(X \geq i)$$

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&= \frac{1}{\binom{n}{k}} \binom{n+1}{k+1} = \frac{n+1}{k+1}
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Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_x x \Pr(X = x \mid E)$$

$$\left(\mathbb{E}[X] = \sum_x x \Pr(X = x) \right)$$

Theorem (A Fourth Way of Computing Expectation (CS 5.6-8))

Let X be a random variable defined on a sample space Ω.

Let $E_1, E_2, \cdots, E_n$ be a partition of Ω.

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid E_i] \Pr(E_i)$$

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$$= \sum_{i=1}^{n} \mathbb{E}[X \mid E_i] \Pr(E_i)$$

($\nexists$) Rational Person Playing a Card Game (CS 5.6 – 4)

A : \$1.00; Repeat

J : \$2.00; End

K : \$3.00; End

Q : \$4.00; End

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Conditioning on the first draw c

$$\mathbb{E}[X] = \frac{1}{4}\Big(\mathbb{E}[X \mid c = A] + \mathbb{E}[X \mid c = J] + \mathbb{E}[X \mid c = K] + \mathbb{E}[X \mid c = Q]\Big)$$

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Conditioning on the first draw c

$$\mathbb{E}[X] = \frac{1}{4}\Big(\mathbb{E}[X \mid c = A] + \mathbb{E}[X \mid c = J] + \mathbb{E}[X \mid c = K] + \mathbb{E}[X \mid c = Q]\Big)$$

$$\mathbb{E}[X \mid c = A] = \mathbb{E}[X] + 1$$

(♯) Rational Person Playing a Card Game (CS 5.6 – 4)

A : \$1.00; Repeat

J : \$2.00; End

K : \$3.00; End

Q : \$4.00; End

Conditioning on the first draw c

$$\mathbb{E}[X] = \frac{1}{4}\Big(\mathbb{E}[X \mid c = A] + \mathbb{E}[X \mid c = J] + \mathbb{E}[X \mid c = K] + \mathbb{E}[X \mid c = Q]\Big)$$

$$\mathbb{E}[X \mid c = A] = \mathbb{E}[X] + 1$$

$$\mathbb{E}[X] = \frac{1}{4}\Big(\mathbb{E}[X] + 1 + 2 + 3 + 4\Big)$$

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$$\mathbb{E}[X \mid c = A] = \mathbb{E}[X] + 1$$

$$\mathbb{E}[X] = \frac{1}{4}\Big(\mathbb{E}[X] + 1 + 2 + 3 + 4\Big) = \frac{10}{3}$$

In-class Exercise: Coin Pattern (Provided by Yifan Pei)

X : # of tosses to get 3 consecutive heads (HHH)

$$\mathbb{E}[X]$$

In-class Exercise: Coin Pattern (Provided by Yifan Pei)

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Conditioning on the first 3 tosses

In-class Exercise: Coin Pattern (Provided by Yifan Pei)

X : # of tosses to get 3 consecutive heads (HHH)

$$\mathbb{E}[X]$$

Conditioning on the first 3 tosses

$T, \quad HT, \quad HHT, \quad HHH$

In-class Exercise: Coin Pattern (Provided by Yifan Pei)

X : # of tosses to get 3 consecutive heads (HHH)

$$\mathbb{E}[X]$$

Conditioning on the first 3 tosses

$$T, \quad HT, \quad HHT, \quad HHH$$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

In-class Exercise: Coin Pattern (Provided by Yifan Pei)

X : # of tosses to get 3 consecutive heads (HHH)

$$\mathbb{E}[X]$$

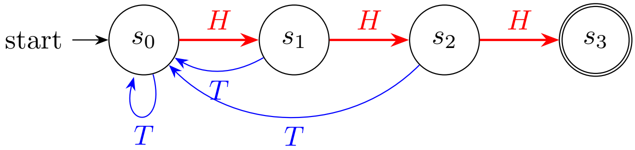
Conditioning on the first 3 tosses

$T, \quad HT, \quad HHT, \quad HHH$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X]+1) + \frac{1}{4}(\mathbb{E}[X]+2) + \frac{1}{8}(\mathbb{E}[X]+3) + \frac{1}{8} \times 3 = 14$$

X : # of tosses to get HHH

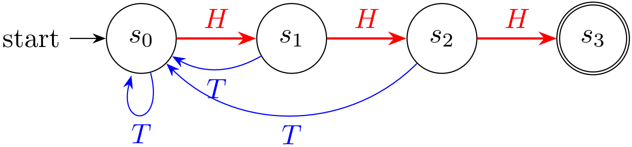
$$T, \quad HT, \quad HHT, \quad HHH$$



$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X]+1) + \frac{1}{4}(\mathbb{E}[X]+2) + \frac{1}{8}(\mathbb{E}[X]+3) + \frac{1}{8} \times 3$$

X : # of tosses to get HHH

$T, \quad HT, \quad HHT, \quad HHH$



$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

$$\mathbb{E}[X_{H^n}] = \cdots = 2(2^n - 1)$$

X : # of tosses to get HHT

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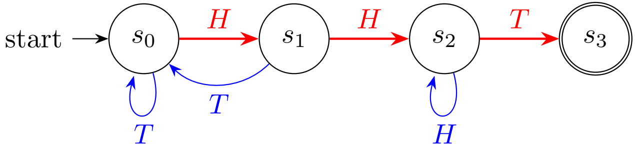
$$\mathbb{E}[X_{HHH}] \text{ } vs. \text{ } \mathbb{E}[X_{HHT}]$$

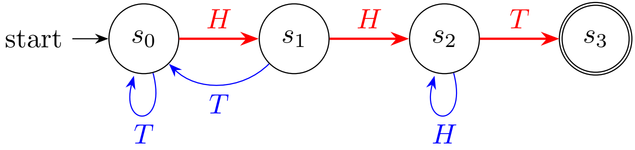
X : # of tosses to get HHT

$$\mathbb{E}[X_{HHH}] \ vs. \ \mathbb{E}[X_{HHT}]$$



X : # of tosses to get HHT

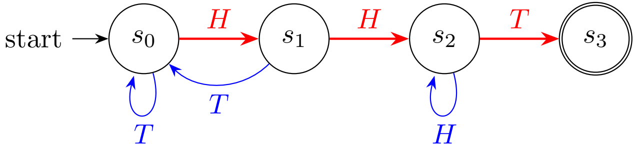
$$\mathbb{E}[X_{HHH}] \; vs. \; \mathbb{E}[X_{HHT}]$$



$$T, \quad HT, \quad HHH, \quad HHT$$

X : # of tosses to get HHT

$$\mathbb{E}[X_{HHH}] \ vs. \ \mathbb{E}[X_{HHT}]$$



$$T, \quad HT, \quad HHH, \quad HHT$$

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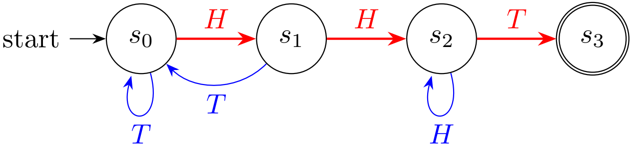
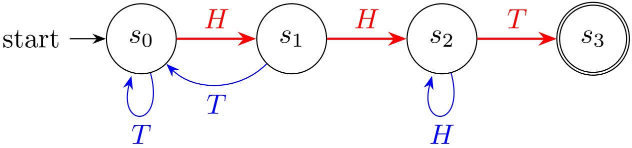
X : # of tosses to get HHT

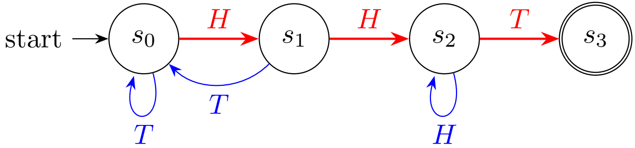
$$\mathbb{E}[X_{HHH}] \ vs. \ \mathbb{E}[X_{HHT}]$$



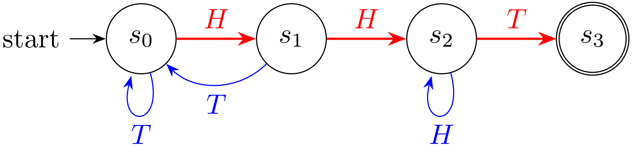
$$T, \quad HT, \quad HHH, \quad HHT$$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$


start
s_0
s_1
s_2
s_3
H
H
T
T
T
H

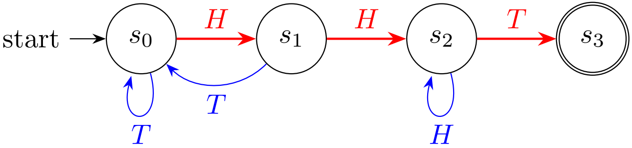


S_i : Expected number of tosses from state s_i to reach state s_n



S_i : Expected number of tosses from state s_i to reach state s_n

$$S_0 = \frac{1}{2}(1 + S_0 + 1 + S_1)$$
$$S_1 = \frac{1}{2}(1 + S_0 + 1 + S_2)$$
$$S_2 = \frac{1}{2}(1 + S_2 + 1 + S_3)$$
$$S_3 = 0$$



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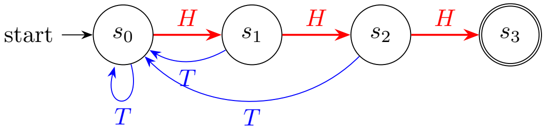
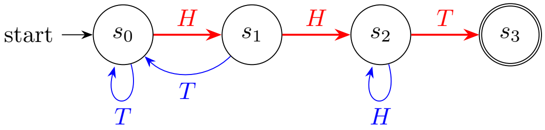
$$S_3 = 0$$

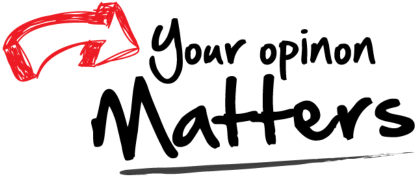
$$S_0 = 8$$





$$\mathbb{E}[X_{HHH}] = 14 > \mathbb{E}[X_{HHT}] = 8$$


Thank
You!



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