# 计算机问题求解－论题2-2
# - 组合与计数

2019年03月04日

# Part I

# 计数与算法

问题1：

讨论算法时间代价时，
我们数什么？

插入排序，也就是“冒泡”

```
(1)  for i = 1 to n − 1
(2)      for j = i + 1 to n
(3)          if (A[i] > A[j])
(4)              exchange A[i] and A[j]
```

How many times is the comparison $A[i] > A[j]$ made in Line 3?

顺便问一句: 这两重循环各自的循环不变量是什么?

critical operation

**Principle 1.1 (Sum Principle)**

The size of a union of a family of mutually disjoint finite sets is the sum of the sizes of the sets.

$$\left|\bigcup_{i=1}^{m} S_i\right| = \sum_{i=1}^{m} |S_i|.$$

# 问题2：你如何理解这里所体现的“抽象”过程？

我们究竟是在数什么？

# 你能解释一下抽象的过程吗？

矩阵相乘

```
(1)  for i = 1 to r
(2)      for j = 1 to m
(3)          S = 0
(4)          for k = 1 to n
(5)              S = S + A[i,k] * B[k,j]
(6)          C[i,j] = S
```

How many multiplications (expressed in terms of $r$, $m$, and $n$) does this pseudocode carry out in total among all the iterations of Line 5?

问题3:

多少集合，

什么关系？

$$T_i = \bigcup_{j=1}^{m} S_j.$$

$$|T_i| = \left|\bigcup_{j=1}^{m} S_j\right| = \sum_{j=1}^{m} |S_j| = \sum_{j=1}^{m} n = mn.$$

# 分块计数

The first loop makes n(n + 1)/2−1 Comparisons.

Ask yourself first where the n(n + 1)/2 comes from and then why we subtracted 1

❶

```
(1)   for i = 1 to n − 1
(2)         minval = A[i]
(3)         minindex = i
(4)         for j = i to n
(5)               if (A[j] < minval)
(6)                     minval = A[j]
(7)                     minindex = j
(8)         exchange A[i] and A[minindex]
```

❷

```
(9)   bigjump = 0
(10)  for i = 2 to n
(11)        if (A[i] > 2 * A[i − 1])
(12)              bigjump = bigjump + 1
```

❶就是排序

❷数“大间隔”

How many comparisons does the pseudocode make in Lines 5 and 11?

# 操作计数与子集计数

**相同的情况，不同的抽象：**

在排序的例子里，对任意含两个元素的子集，我们做一次比较，则比较次数等于$n$个元素的集合所有的两个元素的子集的个数。

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

注意: 数操作实际上是数“有序对”，而子集内元素是无序的。

# 多少种密码?

A password for a certain computer system is supposed to be between four and eight characters long and composed of lowercase and/or uppercase letters. How many passwords are possible? What counting principles did you use? Estimate the percentage of the possible passwords that have exactly four letters.

**问题4:**

**几个集合？相加？相乘？**

# 问题5:

# 每个集合大小怎么算?

$$52^4 + 52^5 + 52^6 + 52^7 + 52^8.$$

你能将这个计算推广到一般的原理吗?

# 从数list到数函数

**Principle 1.4 (Product Principle, Version 2)**

If a set $S$ of lists of length $m$ has the properties that

1. there are $i_1$ different first elements of lists in $S$, and
2. for each $j > 1$ and each choice of the first $j-1$ elements of a list in $S$, there are $i_j$ choices of elements in position $j$ of those lists,

then there are $i_1 i_2 \cdots i_m = \prod_{k=1}^{m} i_k$ lists in $S$.

**问题6：通俗地说说这是什么意思？**

**问题7：这与数函数有什么关系？**

# 平面上 $n$ 个点能生成多少三角形

```
(1)  trianglecount = 0
(2)  for i = 1 to n
(3)       for j = i+1 to n
(4)            for k = j+1 to n
(5)                 if points i, j, and k are not collinear
(6)                      trianglecount = trianglecount + 1
```

Among all iterations of line 5 of the pseudocode, what is the total number of times this line checks three points to see if they are collinear?

前面排序算法可以通过数子集个数来计数，有什么启发吗?

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$$

## 问题8：你能解释这个原理的应用吗？

```
(1)  trianglecount = 0
(2)  for i = 1 to n
(3)      for j = i+1 to n
(4)          for k = j+1 to n
(5)              if points i, j, and k are not collinear
(6)                  trianglecount = trianglecount + 1
```

Among all iterations of line 5 of the pseudocode, what is the total number of times this line checks three points to see if they are collinear?

**Principle 1.5 (Bijection Principle)**

Two sets have the same size if and only if there is a one-to-one function from one set onto the other.

## 问题9：

你能否简单叙述一下：为什么在$n$个数中任取3个不同的数构成的递增序列的集合与所有3个数字构成的子集的集合是等势的？

双射：$(i, j, k) \to \{i, j, k\}$

# Part II
# 利用等价关系计数

## 问题10：

你还记得什么是等价关系吗？它和集合分划有什么关系？

# 等价关系用于计数

问题：
用2种颜色给5个对象着色，并保证每种颜色最少用于2个对象，有多少种不同的着色法？

5个对象不同的排列共有5！种，其中有多少中恰好对应于同一种着色方案？

可以考虑什么样的等价关系，使得等价类的个数就是不同的着色方案的个数？

在这里对称因素起什么作用？

# 给出一个等价类

$\{A, B, C, D, E\}$. Consider the particular labeling in which $A$, $B$, and $D$ are labeled blue and $C$ and $E$ are labeled red. Which lists correspond to this labeling? They are

$$\begin{matrix} ABDCE & ABDEC & ADBCE & ADBEC & BADCE & BADEC \\ BDACE & BDAEC & DABCE & DABEC & DBACE & DBAEC, \end{matrix}$$

$$q \cdot 12 = 120,$$

注意: $q = \binom{5}{3} = 10$

For integers $n$ and $k$ with $0 \le k \le n$, the number of $k$-element subsets of an $n$-element set is

$$\frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}.$$

# 问题11:

# 你能否用对称原理来解释上面的结论?

# Multiset – 有什么不同?

How many *k*-element multisets can we choose from an *n*-element set?

问题12:

为什么一般子集的公式不适用?

Permutation和子集如何对应?

# 等价问题

## 问题13：

Explain how placing $k$ identical books onto the $n$ shelves of a bookcase can be thought of as giving us a $k$-element multiset of the shelves of the bookcase. Explain how distributing $k$ identical apples to $n$ children can be thought of as giving us a $k$-element multiset of the children.

# 书架上的排列问题

暂且假设每本不同

We have $k$ books to arrange on the $n$ shelves of a bookcase. The order in which the books appear on a shelf matters, and each shelf can hold all the books. We will assume that as the books are placed on the shelves, they are pushed as far to the left as they will go. Thus, all that matters is the order in which the books appear. When book $i$ is placed on a shelf, it can go between two books already there or to the left or right of all the books on that shelf.

$$n(n+1)(n+2)\cdots(n+k-1) = \prod_{i=1}^{k}(n+i-1)$$

$$n^{\overline{k}}$$

$$= \prod_{j=0}^{k-1}(n+j) = \frac{(n+k-1)!}{(n-1)!}.$$

## 问题14：

你能否利用等价关系的概念来解释上述结果中的商式？

$$\frac{(n+k-1)!}{(n-1)!}$$

# 回到Multiset问题

对应到书架问题：*k*本书是一样的，所以任何一种排列等完全等价。

The number of $k$-element multisets chosen from an $n$-element set is

$$\frac{n^{\overline{k}}}{k!} = \binom{n+k-1}{k}.$$

# 问题15：

为什么我们在算法分析中经常用到“Σ”？

累加经常不简单！

# 试试看

计算： $\sum_{i=1}^{k} i2^i$

For your reference : Arithmetic - Geometric Series

$$\sum_{i=1}^{k} i2^i = \sum_{i=1}^{k} i\left(2^{i+1} - 2^i\right)$$

$\Sigma$

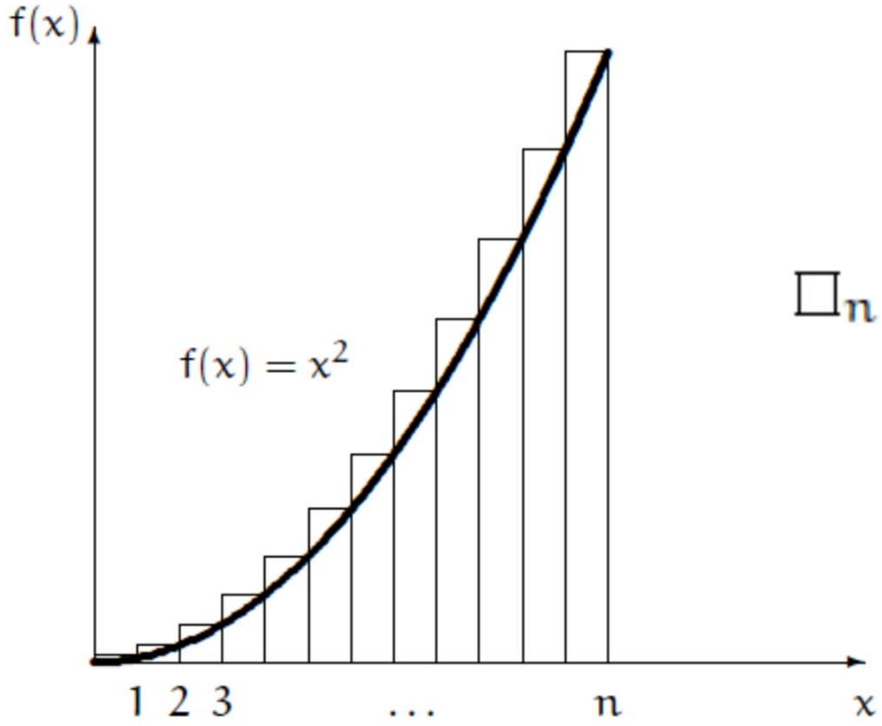
$$= (2^2 + 2\cdot 2^3 + 3\cdot 2^4 + \cdots + (k-1)\cdot 2^k + k\cdot 2^{k+1})$$

$$- (2 + 2\cdot 2^2 + 3\cdot 2^3 + \cdots + (k-1)\cdot 2^{(k-1)} + k\cdot 2^k)$$

$$= (k\cdot 2^{k+1} - 2) - \sum_{i=2}^{k} 2^i = (k\cdot 2^{k+1} - 2) - (2^{k+1} - 4)$$

$$= (k-1)\cdot 2^{k+1} + 2$$

你还记得吗? $\square_n = \sum_{0 \leqslant k \leqslant n} k^2, \qquad \text{for } n \geqslant 0$




$$\square_n = \frac{n(n+1)(2n+1)}{6}$$

The area under this curve is $\int_0^n x^2\,dx = n^3/3$; therefore we know that $\square_n$ is approximately $\frac{1}{3}n^3$.

# 课外作业

- CS pp.8-: 9, 13
- CS pp.20-: 15
- CS pp.30-: 6, 9, 14
- CS pp.54-: 8, 10, 15