1-12 Partial Order and Lattice

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Feb. 25, 2020

SM Problem 14.44: Consistent Enumerations

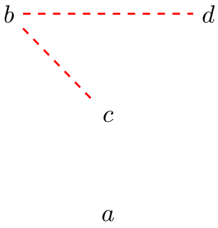
Suppose the following are three consistent enumerations of an ordered set $A = \{a, b, c, d\}$:

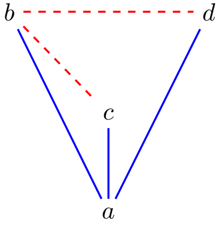
$$\begin{array}{lllll} A_1: & a & b & c & d \\ A_2: & a & c & b & d \\ A_3: & a & c & d & b \end{array}$$

Assuming the Hasse diagram D of A is connected, draw D.

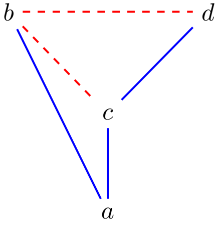
$$b \prec_{A_1} c \land c \prec_{A_2} b \implies b \parallel_A c$$

$$d \prec_{A_3} b \land b \prec_{A_2} d \implies b \parallel_A d$$

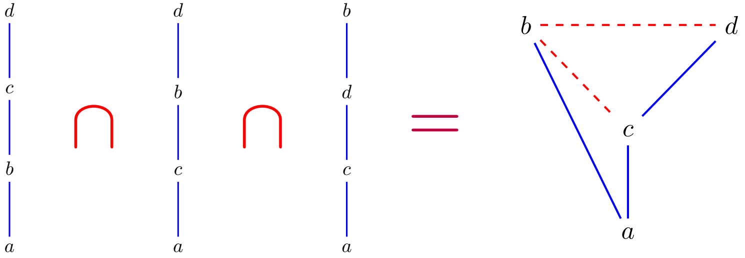




$\# = 6$



$\# = 3$



Theorem

Every partial ordering on a set X is the intersection of the total orders on X containing it.

SM Problem 14.62: Isomorphic Well-Ordered Sets

Suppose A and B are well-ordered isomorphic sets. Show that there is only one isomorphic mapping $f : A \to B$.

$$\text{Well-ordered} \implies \text{Totally-ordered}$$

$$(\mathbb{N}, <)$$

$$\text{Totally-ordered} \not\Longrightarrow \text{Well-ordered}$$

$$(\mathbb{Z}, <)$$

Q : What about "totally-ordered" isomorphic sets?

SM Problem 14.62: Isomorphic Well-Ordered Sets

Suppose A and B are well-ordered isomorphic sets. Show that there is only one isomorphic mapping $f : A \to B$.

$$f : A \to B$$

Make use of the "well-ordered" property.

$$a \leftarrow \min A \qquad b \leftarrow \min B$$

$$f(a) = b$$

$$f\left(\min(A \setminus \{a\})\right) = \min(B \setminus \{b\})$$

$$\boxed{f(x) = \min\Big(B \setminus f(\{a \in A : a < x\})\Big)}$$

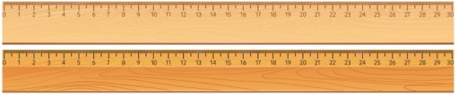
$$\boxed{f(x) = \min\Big(B \setminus f(\{a \in A : a < x\})\Big)}$$

$$f : A \xrightarrow[onto]{1-1} B$$

f is unique

For any isomorphic mapping $g : A \to B$, we show that $g = f$.

$$\boxed{f(x) = \min \Big(B \setminus f(\{a \in A : a < x\}) \Big)}$$



Theorem (Mathematical Induction for Well-Ordered Sets)

Let $\mathcal{S} = (S, <)$ be a well-ordered set. If $P(x)$ is a predicate such that

1. $P(\min S)$ *holds,*
2. $(\forall y < x : P(y)) \implies P(x)$,

then $\forall x \in S : P(x)$.

$$\boxed{f(x) = \min\Big(B \setminus f(\{a \in A : a < x\})\Big)}$$

We need to prove $\forall x \in A : g(x) = f(x)$.

By induction on the structure of A.

$$f(x) = \min \Big(B \setminus f(\{a \in A : a < x\}) \Big)$$

Base Case: Consider $a \leftarrow \min A$.

We need to show that $g(a) = f(a) = b$.

Suppose by contradiction that $g(a) = b_1 \neq b$.

$$\exists a_1 > a : g(a_1) = b < b_1 = g(a)$$

$$f(x) = \min\Big(B \setminus f(\{a \in A : a < x\})\Big)$$

Induction Hypothesis: $\forall y < x : g(y) = f(y)$

Induction Step: We need to show that $g(x) = f(x)$.

Suppose by contradiction that $g(x) \neq f(x)$.

$$f(x) = \min\Big(\cdot\Big) \triangleq M$$

$$g(x) > f(x) = M$$

$$\exists x_1 > x : g(x_1) = M = f(x) < g(x)$$

Definition (Lattice)

A *lattice* is an algebra $\mathcal{L} = (L, \wedge, \vee)$ satisfying,

$$\forall a, b, c \in L,$$

Idempotency:

$$a \wedge a = a \quad a \vee a = a$$

Commutativity:

$$a \wedge b = b \wedge a \quad a \vee b = b \vee a$$

Associativity:

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \quad (a \vee b) \vee c = a \vee (b \vee c)$$

Absorption:

$$a \wedge (a \vee b) = a \quad a \vee (a \wedge b) = a$$

$$a \wedge (a \vee b) = a \quad a \vee (a \wedge b) = a$$

(1) Very useful in lattice computations

$$a \wedge a = a \wedge (a \vee (a \wedge b)) = a$$

(2) The only laws connecting $\wedge$ and $\vee$

$\wedge$-semilattice $\qquad$ $\vee$-semilattice

(3) Ensure that $\wedge$ and $\vee$ induce the same order on L

$$a \leq b \iff a \wedge b = a$$

$$a \leq b \iff a \vee b = b$$

$$a \wedge b = a \iff a \vee b = b$$

SM Problem 14.72: "Weak" Distributive Laws

Prove that for any lattice L:

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$$

$$a \vee (b \wedge c) \leq a \vee b$$

$$a \vee (b \wedge c) \leq a \vee c$$

$$\frac{a \leq b \qquad c \leq d}{(a \vee c) \leq (b \vee d)}$$

Thank You!