问题讨论

2013/12/20

Problem 13.3

Which of the following are functions? Give reasons for your answers.

(a) Define f on $\mathbb{R}$ by $f = \{(x, y) : x^2 + y^2 = 4\}$.

(b) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/(x + 1)$.

(c) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x + y$.

(d) The domain of f is the set of all closed intervals of real numbers of the form $[a, b]$, where $a, b \in \mathbb{R}$, $a \le b$, and f is defined by $f([a, b]) = a$.

(e) Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by $f(n, m) = m$.

(f) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ x & \text{if } x \leq 0 \end{cases} .$$

(g) Define $f : \mathbb{Q} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x+1 & \text{if } x \in 2\mathbb{Z} \\ x-1 & \text{if } x \in 3\mathbb{Z} \\ 2 & \text{otherwise} \end{cases} .$$

(h) The domain of f is the set of all circles in the plane $\mathbb{R}^2$ and, if c is such a circle, define f by $f(c) =$ the circumference of c.

(i) *(For students with a background in calculus.)* The domain of f is the set of all polynomials with real coefficients, and f is defined by $f(p) = p'$. (Here p' is the derivative of p.)

(j) *(For students with a background in calculus.)* The domain of f is the set of all polynomials and f is defined by $f(p) = \int_0^1 p(x)\, dx$. (Here $\int_0^1 p(x)\, dx$ is the definite integral of p.)

Let A and B be sets. A **function** f from A to B is a relation from A to B satisfying

(*i*) for all $a \in A$, there exists $b \in B$ such that $(a, b) \in f$, and

(*ii*) for all $a \in A$, and all $b, c \in B$, if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$.

Problem 13.4.

Let $f : \mathcal{P}(\mathbb{R}) \to \mathbb{Z}$ be defined by

$$f(A) = \begin{cases} \min(A \cap \mathbb{N}) & \text{if } A \cap \mathbb{N} \neq \emptyset \\ -1 & \text{if } A \cap \mathbb{N} = \emptyset \end{cases}.$$

Prove that f above is a well-defined function.

Well-ordering principle of $\mathbb{N}$.

Every nonempty subset of the natural numbers contains a minimum.

Problem 13.5.

Let X be a nonempty set and let A be a subset of X. We define the **characteristic function** of the set A by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in X \setminus A \end{cases}.$$

(a) Since this is called the characteristic function, it probably is a function, but check this carefully anyway.

(b) Determine the domain and range of this function. Make sure you look at all possibilities for A and X.

Problem 13.13.

Let X be a nonempty set. Find all relations on X that are both equivalence relations and functions.

Problem 14.8.

For each of the functions below, determine whether or not the function is one-to-one and whether or not the function is onto. If the function is not one-to-one, give an explicit example to show what goes wrong. If it is not onto, determine the range.

(a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/(x^2 + 1)$.

(b) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin(x)$. (Assume familiar facts about the sine function.)

(c) Define $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $f(n, m) = nm$.

(d) Define $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $f((x, y), (u, v)) = xu + yv$. (Do you recognize this function?)

(e) Define $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $f((x, y), (u, v)) = \sqrt{(x - u)^2 + (y - v)^2}$. (Do you recognize this function?)

(f) Let A and B be nonempty sets and let $b \in B$. Define $f : A \to A \times B$ by $f(a) = (a, b)$.

(g) Let X be a nonempty set, and $\mathcal{P}(X)$ the power set of X. Define $f : \mathcal{P}(X) \to \mathcal{P}(X)$ by $f(A) = X \setminus A$.

(h) Let B be a fixed proper subset of a nonempty set X. Define a function $f : \mathcal{P}(X) \to \mathcal{P}(X)$ by $f(A) = A \cap B$.

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

Problem 14.12.

Let $a, b, c,$ and d be real numbers with $a < b$ and $c < d$. Define a bijection from the closed interval $[a, b]$ onto the closed interval $[c, d]$ and prove that your function is a bijection.

复习

- 人解题和计算机解题
  - Polya’s Four Steps for Problem Solving
- 计算机解题与数学
  - 精确
  - 形式化
  - 分析结果
- 推理
- 常用的证明方法（反证，归纳）

- 基本的算法结构
  - 如何确定循环过程是正确的（循环不变式）
  - 顺序、分支、循环、递归
- 高级程序设计语言的抽象
  - 数据抽象
  - 控制抽象
- 不同的程序设计风格
- 集合及其运算
- 函数