4-11 P and NP

Hengfeng Wei

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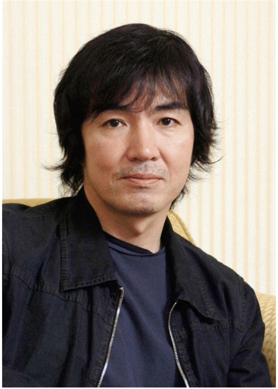
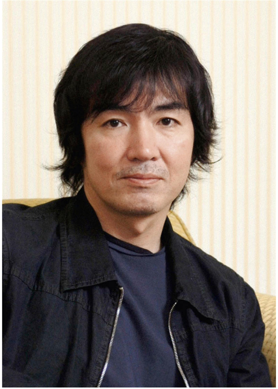
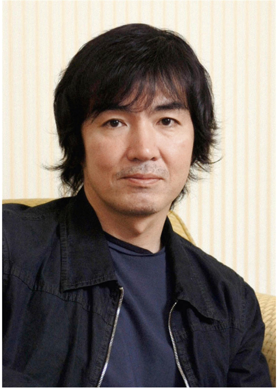
May 20, 2019






东野圭吾
嫌疑人X的献身



“对于数学问题，自己想出解答，
和判断别人说的解答是否正确，何者比较简单”


CONCEPT


decide.

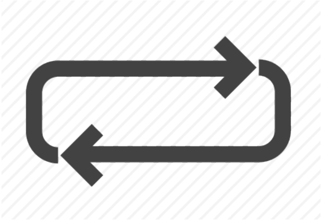

ACCEPT


decide.


ACCEPT


Yes
No
I·DECIDE

Always terminate.



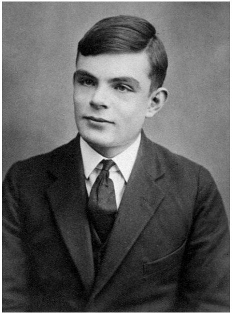
Always terminate.

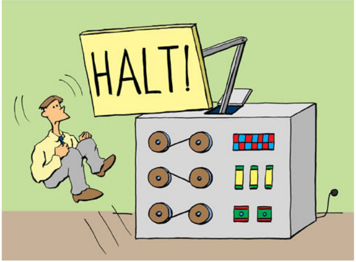
May loop forever for “NO” instance.

Definition (Halting Problem)

Input: An arbitrary program and input

Output: Will the program eventually halt?





Alan designed the perfect computer

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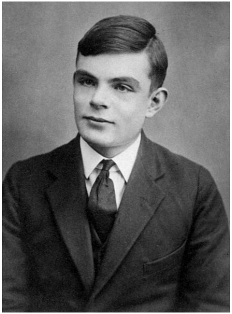

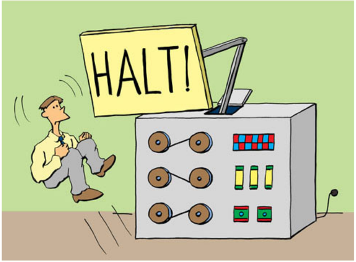

Alan designed the perfect computer

Undecidable

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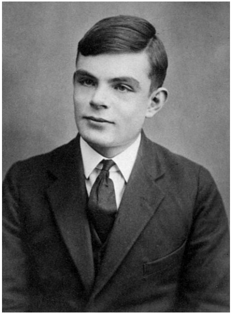

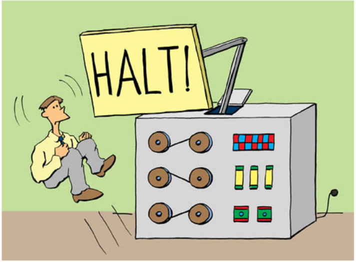

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Undecidable

But Acceptable (Semi-decidable)

$$\mathrm{P} = \Big\{L : L \text{ is decided by a poly. time algorithm}\Big\}$$

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Theorem (Theorem 34.2)

$$P = \Big\{ L : L \textit{ is } \textcolor{red}{\textit{accepted}} \textit{ by a poly. time algorithm} \Big\}$$

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Theorem (Theorem 34.2)

$$P = \Big\{ L : L \textit{ is } \textcolor{red}{\textit{accepted}} \textit{ by a poly. time algorithm} \Big\}$$

You can safely forget "semi-decidable" in computational complexity theory.

Definition (NP)

$$L \in \mathrm{NP}$$

$$\iff$$

$\exists$ poly. time *verifier* $V(x, c)$ such that

$$\forall x \in \{0,1\}^* : x \in L \iff \exists c \text{ with } |c| = O(|x|^k), V(x,c) = 1.$$

NP-problems has short certificates that are easy to verify.

$$\exists L : L \notin \text{NP?}$$

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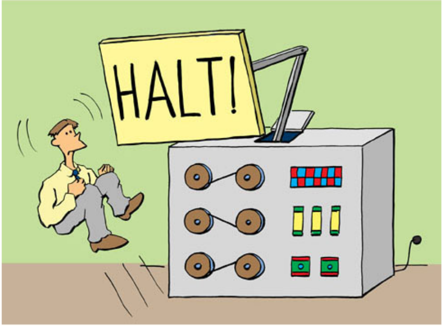

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$\exists L : L \notin \mathrm{NP} \wedge L$ is decidable?

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Theorem (Deterministic Time Hierarchy Theorem (1965))

$$f(n) \log f(n) = o\big(g(n)\big) \implies \mathit{DTIME}\big(f(n)\big) \subsetneq \mathit{DTIME}\big(g(n)\big)$$

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Theorem (Non-deterministic Time Hierarchy Theorem (Cook, 1972))

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$$\mathrm{NP} \subsetneq \mathrm{NEXP}$$

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"Equivalence of Regular Expressions with Squaring" is NEXP-complete:

$$e_1 \cup e_2, \quad e_1 \cdot e_2, \quad e^2$$

Closure of NP (CLRS 34.2-4)

NP is closed under $\cup, \cap, \cdot, \star$.

$$L_1 \in \text{NP}, L_2 \in \text{NP} \implies L = L_1 \circ L_2 \in \text{NP}$$

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procedure V(x, c)

  if $c \neq c_1 \# c_2$ **then**

    return 0

  return $V(x, c_1) \lor V(x, c_2)$

$$L_1 \in \mathrm{NP}, L_2 \in \mathrm{NP} \implies L = L_1 \cup L_2 \in \mathrm{NP}$$

```
1: procedure V(x, c)
2:     if c ≠ c1#c2 then
3:         return 0

4:     return V(x, c1) ∨ V(x, c2)
```

$$x \in L_1 \cup L_2 \iff \exists c, V(x, c) = 1$$

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```

$$x \in L_1 \cap L_2 \iff \exists c, V(x, c) = 1$$

$$L_1 \in \mathrm{NP}, L_2 \in \mathrm{NP} \implies L = L_1 \cdot L_2 \in \mathrm{NP}$$

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```
procedure V(x, c)
    if c ≠ c1#c2&m then
        return 0

    return V(x_{1...m}, c1) ∧ V(x_{m+1...|x|}, c2)
```

$$\textbf{procedure } \mathrm{V}(x, c)$$
$$\quad \textbf{if } c \neq c_1 \# c_2 \& m \textbf{ then}$$
$$\qquad \textbf{return } 0$$
$$\quad \textbf{return } V(x_{1\dots m}, c_1) \land V(x_{m+1\dots|x|}, c_2)$$

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1: procedure V(x, c)
2:     for k ← 1 to |x| do
3:         m_0 ← 0, m_k ← |x|
4:         if c = c_1#c_2#···#c_k&m_1&m_2&···&m_{k-1} then
5:             return ⋀_{i=1}^{i=k} V(x_{m_{i-1}+1...m_i}, c_i)
```

procedure $\mathrm{V}(x, c)$

for $k \leftarrow 1$ **to** $|x|$ **do**

$m_0 \leftarrow 0, m_k \leftarrow |x|$

if $c = c_1 \# c_2 \# \cdots \# c_k \& m_1 \& m_2 \& \cdots \& m_{k-1}$ **then**

return $\bigwedge_{i=1}^{i=k} V(x_{m_{i-1}+1 \ldots m_i}, c_i)$

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```

$$x \in L^{\star} \iff \exists c, A(x, c) = 1$$

Definition (Polynomial-time Reduction)

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$$\forall x : x \in L_1 \iff f(x) \in L_2.$$

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$$\forall L \in \text{NP}, L \leq_p L' \implies L' \text{ is NP-hard}$$

$$\text{NP-complete} = \text{NP} \cap \text{NP-hard}$$

CLRS 34.5-6

HAM-PATH is NP-complete

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HAM-CYCLE $\leq_p$ HAM-PATH

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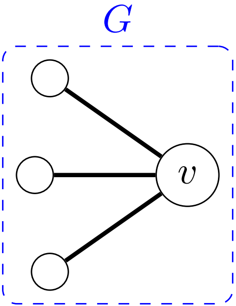

$$G \in \text{HAM-CYCLE} \iff G' \in \text{HAM-PATH}$$

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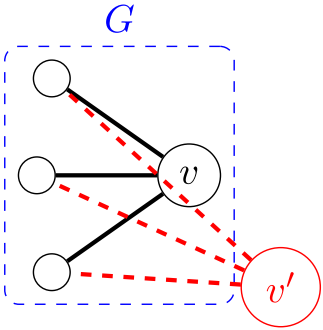

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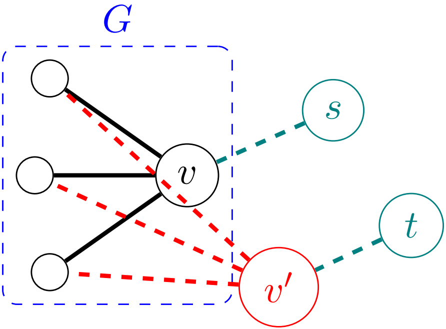

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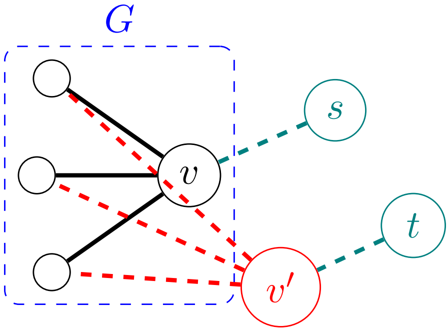

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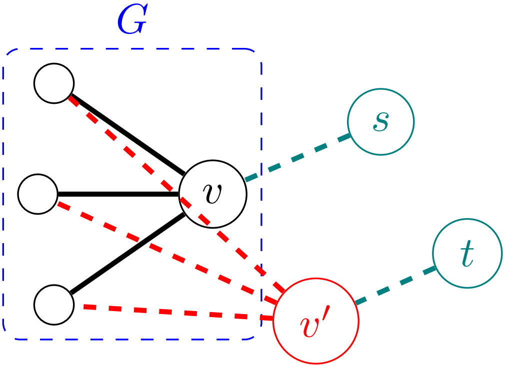
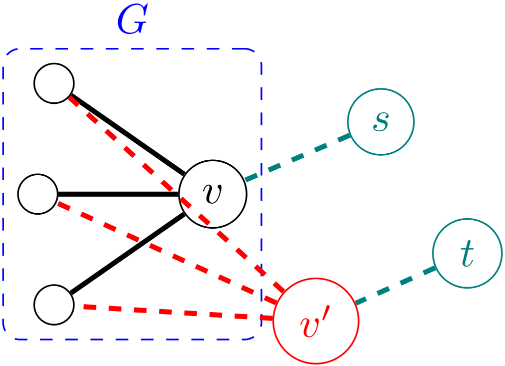
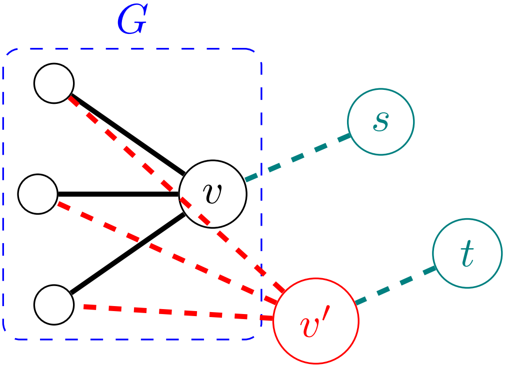


G
v
s
t
v'



$\deg(v) \geq 2$



$\deg(v) \geq 2$

$\forall u \in V(G) : \deg(u) \geq 2$

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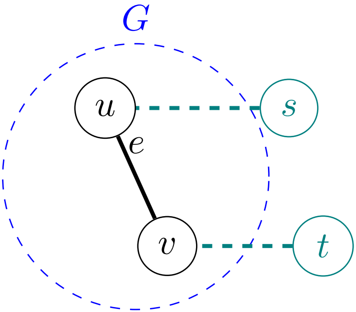
$\forall e \in G$: Construct G_e



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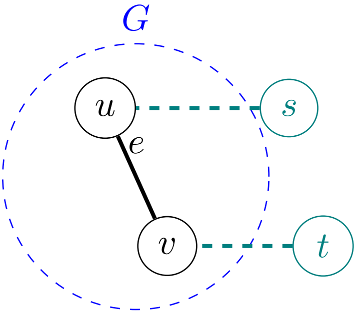

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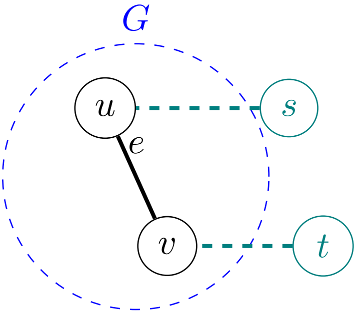

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Karp Reduction

Richard M. Karp (1935 ~)

REDUCIBILITY AMONG COMBINATORIAL PROBLEMS

Richard M. Karp

University of California at Berkeley

(1972)

Cook Reduction

Stephen Cook (1939 $\sim$)

The Complexity of Theorem-Proving Procedures

Stephen A. Cook

University of Toronto

(1971)

$$\text{UNSAT} = \Big\{ \varphi : \varphi \text{ is unsatisfiable.} \Big\}$$

Q : Is UNSAT NP-hard?

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Proof.

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Proof.

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□

$$\boxed{\forall x : x \in L_1 \iff f(x) \in L_2}$$

Theorem (CLRS 34.5-2)

0/1 *ILP is NP-complete.*

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$$\text{3-CNF-SAT} \leq_p 0/1 \text{ ILP}$$

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$x_1 \lor \overline{x_2} \lor \overline{x_3} \iff$

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0/1 ILP is NP-complete.

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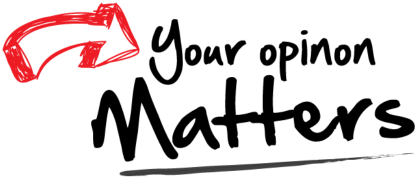
$$\text{3-CNF-SAT} \le_p 0/1 \text{ ILP}$$

$$x_1 \lor \overline{x_2} \lor \overline{x_3} \iff x_1 + (1 - x_2) + (1 - x_3) \ge 1, \quad x_i \in \{0, 1\}$$


THANK YOU



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