Second Isomorphism Theorem

151220102 万子文

Description:

Let H be a subgroup of a group G(not necessarily normal in G) and N a normal subgroup of G.Then HN is a subgroup of G,$H \cap G$ is a normal subgroup of H,and:

$$H/H \cap N \cong HN/N$$

1.HN is a subgroup of G:

Proposition 3.9:

(1)The identity e of G is in H.

(2)If $h_1, h_2 \in H$,then $h_1h_2 \in H$.

(3)If $h \in H$,then $h^{-1} \in H$.

Proof:

(1)$e \in H, e \in N \Rightarrow e \in HN$

(2)Suppose that $h_1n_1, h_2n_2 \in HN$,

$(h_1n_1)(h_2n_2) = h_1h_2(h_2^{-1}n_1h_2)n_2 \in HN$

$(h_1h_2 \in H(h_2^{-1}n_1h_2)n_2 \in N)$

(3)Suppose that $h \in H, n \in N$,

$(hn)^{-1} = n^{-1}h^{-1} = h^{-1}(hn^{-1}h^{-1})$

Since N is normal,$hn^{-1}h^{-1} \in N$,so $(hn)^{-1} \in HN$.

2. $H \cap N$ is normal in H:

Theorem 10.1 For all $g \in G$,$gNg^{-1} \subset N$.

Let $h \in H, n \in N \cap H$,$hnh^{-1} \in H$

Since N is normal in G,$hnh^{-1} \in N$

$\Rightarrow hnh^{-1} \in H \cap N \Rightarrow H \cap N$ is normal in H.

$$H/H \cap N \cong HN/N$$

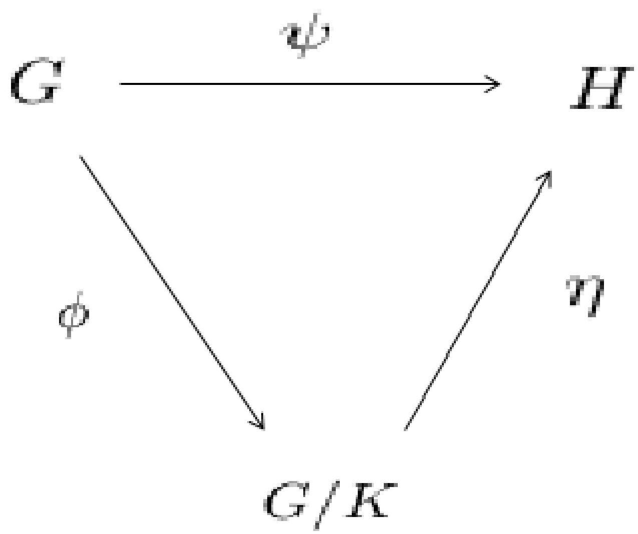
First Isomorphism Theorem If $\psi : G \to H$ is a group homomorphism with $K = ker\psi$,then K is normal in G.Let $\phi : G \to G/K$ be the canonical homomorphism.Then there exists a unique isomorphism $\eta : G/K \to \psi(G)$ such that $\psi = \eta\phi$.

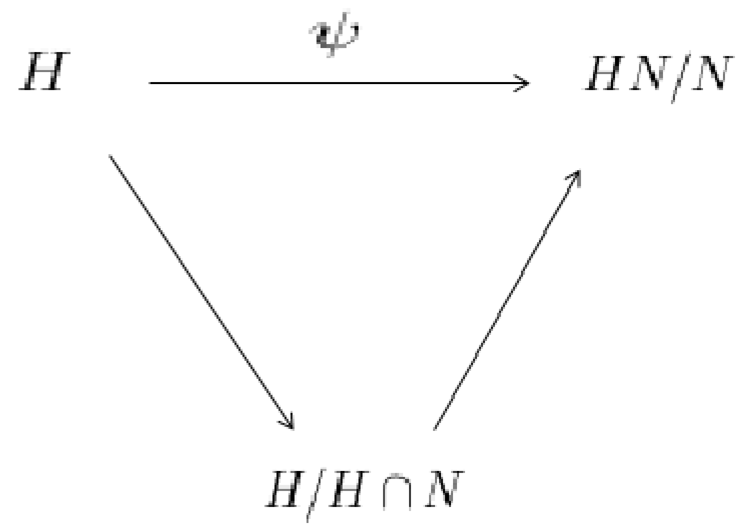
Consider if we have $\psi : H \to HN/N$

(1)ψ is a group homomorphism

(2)$ker\psi = H \cap N$

(3)ψ is onto


ψ
G
H
φ
η
G/K


ψ
H
HN/N
H/H ∩ N

Now we define a map $\psi : H \to HN/N$ by $h \to hN$.

(1)ψ is a homomorphism:

$$\psi(h_1h_2) = h_1h_2N = h_1Nh_2N = \psi(h_1)\psi(h_2)$$

(2)$Ker\psi = \{h \in H, h \in N\} = H \cap N$

(3)The map is onto.Any coset $hnN = hN$ is the image of h in H.

So,we have $HN/N \cong H/H \cap N$.

Thank you