

问题讨论

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Problem 10.2.

Let $X = \{1, 2, 3, 4, 5\}$.

- (a) If possible, define a relation on X that is an equivalence relation.
- (b) If possible, define a relation on X that is reflexive, but neither symmetric nor transitive.
- (c) If possible, define a relation on X that is symmetric, but neither reflexive nor transitive.
- (d) If possible, define a relation on X that is transitive, but neither reflexive nor symmetric.

A relation on a set X is said to be **reflexive** if $x \sim x$ for all $x \in X$.

The relation is **symmetric** if for all $x, y \in X$, whenever $x \sim y$, then $y \sim x$.

Finally, the relation is **transitive** if for all $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

Problem 10.4.

Define a relation \sim on \mathbb{R}^2 as follows: For $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, we say that $(x_1, x_2) \sim (y_1, y_2)$ if and only if both $x_1 - y_1$ and $x_2 - y_2$ are even integers. Is this relation an equivalence relation? Why or why not?

Once we have an equivalence relation on a set X , we define the **equivalence class** of an element $x \in X$ to be the set E_x where $E_x = \{y \in X : x \sim y\}$. Using this notation, we see that the sets

Problem⁴ 10.5.

Let X be a nonempty set with an equivalence relation \sim on it. Prove that for all elements x and y in X , the equality $E_x = E_y$ holds if and only if $x \sim y$.

Problem 10.8.

Recall that a **polynomial** p over \mathbb{R} is an expression of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$ where each $a_j \in \mathbb{R}$ and $n \in \mathbb{N}$. The largest integer j such that $a_j \neq 0$ is the **degree** of p . We define the degree of the constant polynomial $p = 0$ to be $-\infty$. (A polynomial over \mathbb{R} defines a function $p : \mathbb{R} \rightarrow \mathbb{R}$.)

- (a) Define a relation on the set of polynomials by $p \sim q$ if and only if $p(0) = q(0)$. Is this an equivalence relation? If so, what is the equivalence class of the polynomial given by $p(x) = x$?
- (b) Define a relation on the set of polynomials by $p \sim q$ if and only if the degree of p is the same as the degree of q . Is this an equivalence relation? If so, what is E_r if $r(x) = 3x + 5$?
- (c) Define a relation on the set of polynomials by $p \sim q$ if and only if the degree of p is less than or equal to the degree of q . Is this an equivalence relation? If so, what is E_r , where $r(x) = x^2$?

The precise definition of a partition is the following. Let X be a set. Then a family of sets $\{A_\alpha : \alpha \in I\}$ is a **partition of X** if three things happen:

- (i) For every $\alpha \in I$, the set A_α is nonempty,
- (ii) $\bigcup_{\alpha \in I} A_\alpha = X$, and
- (iii) for all $\alpha, \beta \in I$, if $A_\alpha \cap A_\beta \neq \emptyset$, then $A_\alpha = A_\beta$.

Problem 11.9.

Let X be a nonempty set and $\{A_\alpha : \alpha \in I\}$ be a partition of X .

- (a) Let B be a subset of X such that $A_\alpha \cap B \neq \emptyset$ for every $\alpha \in I$. Is $\{A_\alpha \cap B : \alpha \in I\}$ a partition of B ? Prove it or give a counterexample.
- (b) Suppose further that $A_\alpha \neq X$ for every $\alpha \in I$. Is $\{X \setminus A_\alpha : \alpha \in I\}$ a partition of X ? Prove it or give a counterexample.

Let A be a nonempty set of real numbers that is bounded above. Then a real number U is said to be a **supremum** of A or **least upper bound** of A if

- (i) $a \leq U$ for all $a \in A$, and
- (ii) if $M \in \mathbb{R}$ satisfies $a \leq M$ for all $a \in A$, then $U \leq M$.

Problem 12.10.

Let S and T be nonempty bounded subsets of \mathbb{R} .

- (a) Show that $\sup(S \cup T) \geq \sup S$, and $\sup(S \cup T) \geq \sup T$.
- (b) Show that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.
- (c) Try to state the results of (a) and (b) in English, without using mathematical symbols.

Problem 12.13.

Let \sim denote a relation on a set S . The relation \sim is called a **partial order** if the following three conditions are satisfied.

- (i) (Reflexive property) For all $x \in S$, we have $x \sim x$.
- (ii) (Transitive property) For all $x, y, z \in S$, if $x \sim y$ and $y \sim z$, then $x \sim z$.
- (iii) (Antisymmetric property) For all $x, y \in S$, if $x \sim y$ and $y \sim x$, then $x = y$.

The relation \sim is a **total order** on the set S if, in addition, (iv) below is satisfied.

- (iv) For all $x, y \in S$, either $x \sim y$ or $y \sim x$.
 - (a) Show that the relation $x \sim y$ if and only if $x \leq y$ defines a total order on \mathbb{R} .
 - (b) Let A be a set containing at least two elements. We define an order on $\mathcal{P}(A)$ using the regular set inclusion \subseteq . Show that $(\mathcal{P}(A), \subseteq)$ is a partial order, but not a total order.
 - (c) Consider the relation $<$ on \mathbb{R} . Show that this is not a total order by exhibiting counterexamples for each total order property that is violated.

Problem 12.16.

You showed in Problem 12.13 that $(\mathcal{P}(\mathbb{Z}), \subseteq)$ is a partial order. For every nonempty subset \mathcal{A} of $\mathcal{P}(\mathbb{Z})$ we say that $U \in \mathcal{P}(\mathbb{Z})$ is an upper set of \mathcal{A} , if $X \subseteq U$ for all $X \in \mathcal{A}$. A nonempty set $\mathcal{A} \subseteq \mathcal{P}(\mathbb{Z})$ will be called an upper bounded set if there is an upper set of \mathcal{A} in $\mathcal{P}(\mathbb{Z})$. We say $U_0 \in \mathcal{P}(\mathbb{Z})$ is a least upper set if (i) U_0 is an upper set of \mathcal{A} and (ii) if U is another upper set of \mathcal{A} , then $U_0 \subseteq U$.

- (a) Let $\mathcal{B} = \{\{1, 2, 5, 7\}, \{2, 8, 10\}, \{2, 5, 8\}\}$. Show that \mathcal{B} is an upper bounded set and find a least upper set of \mathcal{B} , if there is one.
- (b) Prove that every nonempty subset of $\mathcal{P}(\mathbb{Z})$ is upper bounded.
- (c) Define “lower set,” “lower bounded set,” and “greatest lower set.”
- (d) Let \mathcal{A} be a nonempty subset of $\mathcal{P}(\mathbb{Z})$. Using union and intersection, find an expression for least upper set of \mathcal{A} and greatest lower set of \mathcal{A} .
- (e) Prove that $(\mathcal{P}(\mathbb{Z}), \subseteq)$ has the “least upper set property” (in other words, show every upper bounded set has a least upper set).

Problem 12.20.

Suppose we define ∞ to be an object that satisfies $a \leq \infty$ for all $a \in \mathbb{R}$. Prove that $\infty \notin \mathbb{R}$.

Problem 12.22.

Prove that if a is a rational number, then there is an irrational number b such that $a < b$.

Problem 12.23.

Prove that for two arbitrary real numbers a and b with $a < b$, there is an irrational number c such that $a < c < b$. (Hint: Consider $a/\sqrt{2}$ and $b/\sqrt{2}$.)

Project 27.4

Theorem 27.3.

There exist irrational numbers a and b such that a^b is rational.

Complete the proof of this theorem, using appropriate choices for a and b and the two cases below:

Case 1. $\sqrt{2}^{\sqrt{2}}$ is a rational number;

Case 2. $\sqrt{2}^{\sqrt{2}}$ is an irrational number.

6. There are many other examples of irrational powers of irrational numbers that are rationals, assuming you know lots of different ways to express irrational numbers. See if you can come up with another example based on the fact that the natural logarithm of 2, denoted $\ln 2$, is irrational. Can you find other examples?

Theorem 27.4.

There exist irrational numbers a and b such that a^b is irrational.

The proof of this theorem is also a proof in cases. We suggest that you figure out how to use $\sqrt{2}^{\sqrt{2}}$ and the number $\sqrt{2}^{(\sqrt{2}+1)}$.

Some discussions

You've seen numbers ever since you've been in school, and you know a lot about them. It is possible to give them a careful mathematical foundation. In fact, it's possible to construct the natural numbers (and you can do so in Project 27.3). Then, if you try to introduce operations like addition and subtraction, you'll find that you are missing something: the negative numbers. So you look at the integers, and try again. Now, trying to introduce multiplication and division, you'll find you are missing something again: multiplicative inverses. So you look at the rational numbers, and you'll find you are missing something yet again. That brings you to the real numbers.