群同构第二定理的证明

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群同构第二定理

Theorem.

Let H be a subgroup of a group G and N a normal subgroup of G. Then

- HN is a subgroup of G
- $H \cap N$ is a normal subgroup of H
- $H/(H \cap N) \cong (HN)/N$

证明

1 $HN \leq G, HN = \{hn : h \in H, n \in N\}$

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1 $HN \leq G, HN = \{hn : h \in H, n \in N\}$

Proof.

1.1 封闭性

Suppose that $h_1 n_1, h_2 n_2 \in HN$, N is normal in G

According to Theorem 10.3

$(h_2)^{-1} n_1 h_2 \in N, ((h_2)^{-1} n_1 h_2) n_2 \in N$

$(h_1 n_1)(h_2 n_2) = h_1 h_2 ((h_2)^{-1} n_1 h_2) n_2$ is in HN

$\square$

Theorem 10.3

Theorem.

Let G be a group and N be a subgroup of G. Then the following statements are equivalent

- $N \trianglelefteq G$
- For all $g \in G, gNg^{-1} \subset N$
- For all $g \in G, gNg^{-1} = N$

证明

1 $HN \leq G, HN = \{hn : h \in H, n \in N\}$

Proof.

1.2 单位元存在

Since $H, N \leq G, e \in H \cap N$

$\forall hn \in HN, (hn)e = e(hn) = hn$

$\square$

证明

1 $HN \leq G, HN = \{hn : h \in H, n \in N\}$

Proof.

1.3 **逆元存在**

$\forall hn \in HN, (hn)^{-1} = n^{-1}h^{-1} = (h^{-1}h)nh^{-1}$

$(h^{-1}h)n^{-1}h^{-1} \in G$ because

$(h^{-1}h)n^{-1}h^{-1} = h^{-1}(hnh^{-1})$ and $h^{-1} \in H, (hnh^{-1}) \in N$

□

证明

2 $H \cap N$ is normal in H

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Proof.

$\forall h \in H, n \in H \cap N$

- $hnh^{-1} \in H$ because $h, n, h^{-1} \in H$

$\square$

证明

2 $H \cap N$ is normal in H

Proof.

$\forall h \in H, n \in H \cap N$

- $hnh^{-1} \in H$ because $h, n, h^{-1} \in H$
- $hnh^{-1} \in N$ because $n \in N$, N is normal in G and $h \in G$

$\square$

证明

2 $H \cap N$ is normal in H

Proof.

$\forall h \in H, n \in H \cap N$

- $hnh^{-1} \in H$ because $h, n, h^{-1} \in H$
- $hnh^{-1} \in N$ because $n \in N$, N is normal in G and $h \in G$
- $hnh^{-1} \in H \cap N$, so $H \cap N$ is normal in H

□

证明

3 $(HN)/N \cong H/(H \cap N)$

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Proof.

- $(HN)/N = \{hnN : hn \in HN\} = \{hN : h \in H\}$

$\square$

证明

3 $(HN)/N \cong H/(H \cap N)$

Proof.

- $(HN)/N = \{hnN : hn \in HN\} = \{hN : h \in H\}$
- define $\phi : H \to (HN)/N$ by $h \mapsto hN$
  the map ϕ is onto: $\forall hN \in (HN)/N, \phi(h) = hN$

$\square$

证明

3 $(HN)/N \cong H/(H \cap N)$

Proof.

define $\phi : H \to (HN)/N$ by $h \mapsto hN$

- ϕ is homomorphism:
  $\forall h_1, h_2 \in H, \phi(h_1 h_2) = h_1 h_2 N = h_1 N h_2 N = \phi(h_1)\phi(h_2)$

$\square$

证明

3 $(HN)/N \cong H/(H \cap N)$

Proof.

define $\phi : H \to (HN)/N$ by $h \mapsto hN$

- Because ϕ is a homomorphism, according to First Isomorphism Theorem

$\square$

群同构第一定理

Theorem.

- If $\phi : H \to (HN)/N$ is a group homomorphism and $K = \ker \phi$, then K is normal in H
- Let $\phi_1 : H \to H/K$ be the canonical homomorphism then there exists a unique isomorphism $\phi_2 : H/K \to \phi(H)$ such that $\phi = \phi_2\phi_1$

证明

3 $(HN)/N \cong H/(H \cap N)$

Proof.

- $K = \ker\phi$, $K \trianglelefteq H$
- $\phi : H \to (HN)/N$ homomorphism
- $\phi_1 : H \to H/K$ canonical homomorphism
- $\phi_2 : H/K \to \phi(H) = (HN)/N$ unique isomorphism
- $\phi = \phi_2\phi_1$

□

证明

3 $(HN)/N \cong H/(H \cap N)$

Proof.

- $HN/N = \phi(H) \cong H/K$

$\square$

证明

3 $(HN)/N \cong H/(H \cap N)$

Proof.

- $(HN)/N = \phi(H) \cong H/K$
- $K = \phi^{-1}(\{eN\}) = \phi^{-1}(\{N\})$
  $K \trianglelefteq H, nN = N$
  $K = H \cap N$

$\square$

证明

3 $(HN)/N \cong H/(H \cap N)$

Proof.

- $(HN)/N = \phi(H) \cong H/K$
- $K = H \cap N$
- $(HN)/N = \phi(H) \cong H/(H \cap N)$

$\square$

问题

Proof.

define a map ϕ from H to HN/N by $H \mapsto hN$. The map ϕ is onto, since any coset $hnN = hN$ is the image of h in H. We also know that ϕ is a homomorphism because

$$\phi(hh') = hh'N = hNh'N = \phi(h)\phi(h')$$

By the First Isomorphism Theorem, the image of ϕ is isomorphic to $H/\ker\phi$:

$$HN/N = \phi(H) \cong H/\ker\phi$$

$\square$

问题

Definition.

A homomorphism between groups $(G, \cdot)$ and $(H, \circ)$ is a map $\phi : G \to H$ such that

$$\phi(g_1 \cdot g_2) = \phi(g_1) \circ \phi(g_2)$$

请问我们是否可以不用证明 the ϕ is onto, 只证明:

- the ϕ is well defined
- and the image of homomorphism is subset of $(HN)/N$?

然后根据 First Isomorphism Theorem 得出 $(HN)/N \cong H/K$

问题

- $(HN)/N \overset{1}{=} \phi(H) \overset{2}{\cong} H/K \overset{3}{=} H/(H \cap N)$