

# 计算机问题求解 – 论题2-5

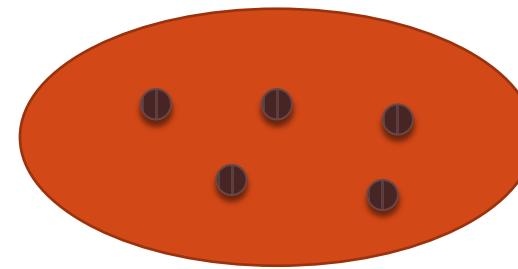
- 离散概率基础

课程研讨

- CS第5章第1-4节

# 问题1： probability

- 你理解这些概念了吗？
  - Sample space
  - Element
  - Event
  - Probability weight
  - Probability
- 你能基于这些概念解释probability distribution function的三个条件吗？
  1.  $P(A) \geq 0$  for any  $A \subseteq S$ .
  2.  $P(S) = 1$ .
  3.  $P(A \cup B) = P(A) + P(B)$  for any two disjoint events  $A$  and  $B$ .



# 问题1： probability (续)

- 在这些例子中， sample space、 element、 event分别是什么？
  - The probability of getting at least 1 head in 5 flips of a coin.
  - The probability of getting a total of 6 or 7 on the 2 dice.
  - The probability that all 3 keys hash to different locations (among 20).
- 你能给出它们的答案吗？

# 问题1： probability (续)

- 你理解uniform probability distribution了吗？

**Theorem 5.2** Suppose  $P$  is the uniform probability measure defined on a sample space  $S$ . Then for any event  $E$ ,

$$P(E) = |E|/|S|,$$

the size of  $E$  divided by the size of  $S$ .

- 现在你能给出之前几题的答案了吗？
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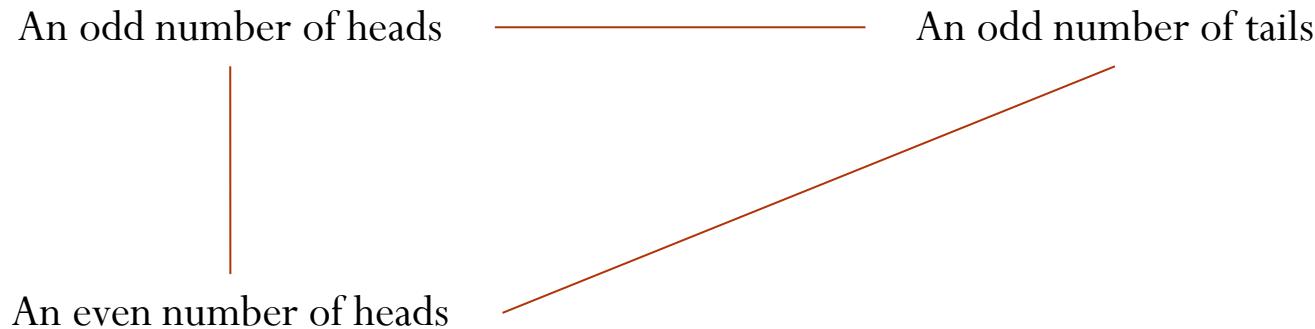
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probability → counting

# 问题1： probability (续)

- What is the probability of an odd number of heads in three tosses of a coin? (假设是uniform probability distribution)
  - 如何利用这个三角形快速求解?



- 如果不是uniform probability distribution, 怎么办?

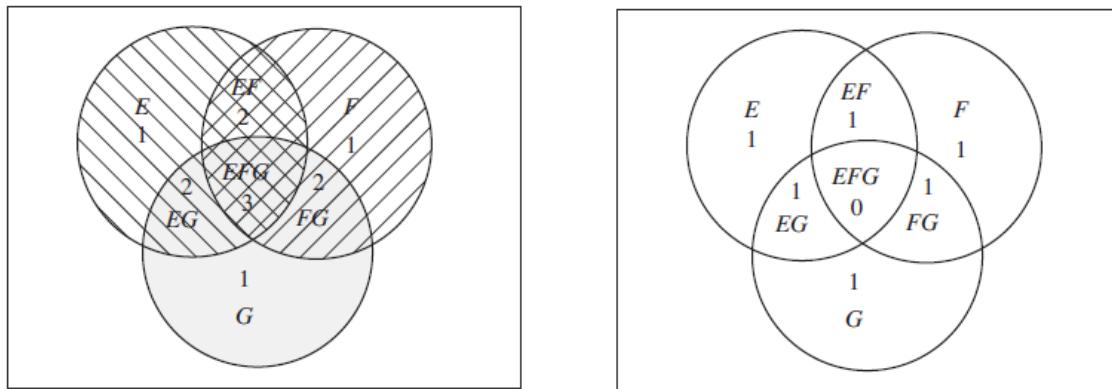
# 问题1： probability (续)

- Which is more likely, or are both equally likely?
  - Drawing an ace and a king when you draw two cards from among the 13 spades, or drawing an ace and a king when you draw two cards from an ordinary deck of 52 playing cards?
  - Drawing an ace and a king of the same suit when you draw two cards from a deck, or drawing an ace and a king when you draw two cards from among the 13 spaces?

## 问题2： the principle of inclusion and exclusion

- 你理解这两个图的含义了吗？

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G)$$



- 你读懂这个公式了吗？

$$P\left(\bigcup_{i=1}^n E_i\right) = \sum_{k=1}^n (-1)^{k+1} \sum_{\substack{i_1, i_2, \dots, i_k: \\ 1 \leq i_1 < i_2 < \dots < i_k \leq n}} P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k})$$

# 交集非空的事件

- 掷均匀的色子，掷3次。出现事件“或者3次均相等，或者没有一次是4”的概率是多少？
- 合理假设：每个outcome出现的可能性是一样的。
- 样本空间大小是  $6^3=216$ 。
- 用  $F$  表示事件“3次结果一样”，则  $|F|=6$  ( $F=\{111,222,\dots,666\}$ )
- 用  $G$  表示事件“没有一次结果是4”，则  $|G|=5^3=125$  ( $G$  是从集合  $\{1,2,3,5,6\}$  中任选3个数的组合数)
- 要求的事件为  $F$  和  $G$  的并集：

$$|F \cup G| = |F| + |G| - |F \cap G| = 6 + 125 - 5 = 126$$

- 因此，最终结果是： $126/216 = 7/12$

## 问题2： the principle of inclusion and exclusion (续)

- How many functions from an  $m$ -element set  $M$  to an  $n$ -element set  $N$  map nothing to at least one element of  $N$ ?
  - Sample space?
  - Element?
  - Event?

$$\left| \bigcup_{i=1}^n E_i \right| = \sum_{k=1}^n (-1)^{k+1} \underbrace{\sum_{\substack{i_1, i_2, \dots, i_k: \\ 1 \leq i_1 < i_2 < \dots < i_k \leq n}} |E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}|}$$

↗ 在这里如何计算？

E<sub>i</sub>是什么？

## 问题2：the principle of inclusion and exclusion (续)

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不映射到*i*的函数集合

## 问题2： the principle of inclusion and exclusion (续)

- In how many ways may you distribute  $k$  identical apples to  $n$  children so that no child gets more than  $m$ ?

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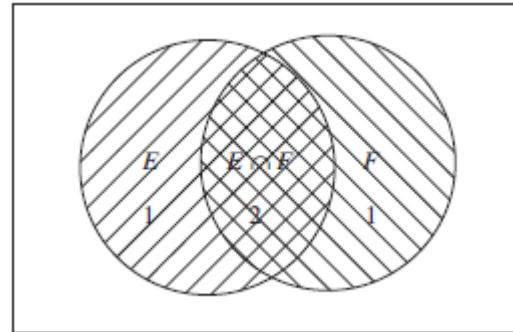
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$$\binom{k + (n - 1)}{n - 1} - \sum_{i=1}^n (-1)^{i+1} \binom{n}{i} \binom{k - (m + 1)i + (n - 1)}{n - 1}$$

# 问题3：conditional probability

- 你能结合Venn图解释条件概率的定义吗？

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$



- 你能结合图解释独立性吗？  
 $P(E|F) = P(E)$
- 你能自己推导出这两个定理吗？

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

Theorem 5.4 Suppose  $E$  and  $F$  are events in a sample space. Then  $E$  is independent of  $F$  if and only if  $P(E \cap F) = P(E)P(F)$ .

# 问题3： conditional probability (续)

$$P(x_i = a_i | x_1 = a_1, \dots, x_{i-1} = a_{i-1}) = P(x_i = a_i)$$

- 你理解independent trials process了吗？

**Exercise 5.3-7** Suppose we draw a card from a standard deck of 52 cards, discard it (i.e. we do not replace it), draw another card and continue for a total of ten draws. Is this an independent trials process?

- 为什么这不是一个independent trials process？
- 为这个过程绘制tree diagram，并计算：第*i*张抽到梅花A的概率是多少？
- 如果是independent trials process，其tree diagram有什么特征？

# 问题3：conditional probability (续)

- A nickel, two dimes, and two quarters are in a cup. We draw three coins, one at a time, without replacement.
  - Draw the probability tree which represents the process.
  - Use the tree to determine the probability of getting a nickel on the last draw.
  - Use the tree to determine the probability that the first coin is a quarter, given that the last coin is a quarter.

# 问题4： random variables

- 你理解这些概念了吗？能自己举个例子吗？

- Random variable
  - Expected value

$$E(X + Y) = E(X) + E(Y)$$

- 你能直观解释它们为什么相等吗？

$$E(X) = \sum_{i=1}^k x_i P(X = x_i)$$

$$E(X) = \sum_{s:s \in S} X(s) P(s)$$

## 问题4： random variables (续)

- How many sixes do we expect to see on top if we roll 24 dice?
- What is the expected number of times we need to roll two dice until we get a 7?

## 问题4： random variables (续)

- A student is taking a true-false test and guessing when he doesn't know the answer. We are going to compute a score by subtracting a percentage of the number of incorrect answers from the number of correct answers. That is, for some number  $y$ , the student's corrected score will be

$$(\text{number of corrected answers}) - y(\text{number of incorrect answers})$$

When we convert this “corrected score” to a percentage score, we want its expected value to be the percentage of the material being tested that the student knows. How can we do this?

# 随机变量的期望值与算法分析

这是什么?它为什么在算法分析中很有用?

```
FindMin(A, n)
    // Finds the smallest element in Array A, where n = |A|
(1) min = A[1]
(2) for i = 2 to n
(3)     if (A[i] < min)
(4)         min = A[i]
(5) return min
```

问题建模:

定义随机变量  $X$ : 赋值语句执行次数  
求  $X$  的期望

定义事件:  $A[i]$  被赋值给  $min$ 。此时  $X$  可以解读为什么?

We solve this problem by letting  $X$  be the number of times that  $min$  is assigned a value and  $X_i$  be the indicator random variable for the event that  $A[i]$  is assigned to  $min$ . Then  $X = X_1 + X_2 + \dots + X_n$ , and  $E(X_i)$  is the probability that  $A[i]$  is the smallest element in the set  $\{A[1], A[2], \dots, A[i]\}$ . Because  $(i - 1)!$  of the  $i!$  permutations of these elements have  $A[i]$  as the smallest element,  $E(X_i) = 1/i$ . Thus,

$$E(X) = \sum_{i=1}^n \frac{1}{i}.$$