A Proof to ***Lemma 29.2***

Lemma 29.2

Given a linear program $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$, suppose that the call to INITIALIZE-SIMPLEX in line 1 of SIMPLEX returns a slack form for which the basic solution is feasible. Then if SIMPLEX returns a solution in line 17, that solution is a feasible solution to the linear program. If SIMPLEX returns “unbounded” in line 11, the linear program is unbounded.

```
SIMPLEX(A, b, c)
1  (N, B, A, b, c, ν) = INITIALIZE-SIMPLEX(A, b, c)
2  let Δ be a new vector of length n
3  while some index j ∈ N has c_j > 0
4      choose an index e ∈ N for which c_e > 0
5      for each index i ∈ B
6          if a_ie > 0
7              Δ_i = b_i / a_ie
8          else Δ_i = ∞
9      choose an index l ∈ B that minimizes Δ_i
10     if Δ_l == ∞
11         return "unbounded"
12     else (N, B, A, b, c, ν) = PIVOT(N, B, A, b, c, ν, l, e)
13 for i = 1 to n
14     if i ∈ B
15         x̄_i = b_i
16     else x̄_i = 0
17 return (x̄_1, x̄_2, ..., x̄_n)
```

At the start of each iteration of the **while** loop of lines 3–12,

1. the slack form is equivalent to the slack form returned by the call of INITIALIZE-SIMPLEX,
2. for each $i \in B$, we have $b_i \geq 0$, and
3. the basic solution associated with the slack form is feasible.

Maintenance:

第一部分:
PIVOT过程返回的松弛型和前一次迭代中的松弛型等价，因此，也与初始松弛型等价。

第二部分:
设在每次迭代开始前，所有的b都大于等于0。

```
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13 for i = 1 to n
14     if i ∈ B
15         x̄_i = b_i
16     else x̄_i = 0
17 return (x̄_1, x̄_2, ..., x̄_n)
```

```
PIVOT(N, B, A, b, c, ν, l, e)
1  // Compute the coefficients of the equation for new basic variable x_e.
2  let Â be a new m × n matrix
3  b̂_e = b_l / a_le
4  for each j ∈ N − {e}
5      â_ej = a_lj / a_le
6  â_el = 1 / a_le
7  // Compute the coefficients of the remaining constraints.
8  for each i ∈ B − {l}
9      b̂_i = b_i − a_ie b̂_e
10     for each j ∈ N − {e}
11         â_ij = a_ij − a_ie â_ej
12     â_il = −a_ie â_el
13 // Compute the objective function.
14 ν̂ = ν + c_e b̂_e
15 for each j ∈ N − {e}
16     ĉ_j = c_j − c_e â_ej
17 ĉ_l = −c_e â_el
18 // Compute new sets of basic and nonbasic variables.
19 N̂ = N − {e} ∪ {l}
20 B̂ = B − {l} ∪ {e}
21 return (N̂, B̂, Â, b̂, ĉ, ν̂)
```

首先，观察到$\hat{b}_e \geq 0$，这是因为根据循环不变式有 $b_l \geq 0$，根据 SIMPLEX 的第 6 行和第 9 行，有 $a_{le} > 0$，根据 PIVOT 的第 3 行，有$\hat{b}_e = b_l / a_{le}$。

SIMPLEX(A, b, c)

```
(N, B, A, b, c, ν) = INITIALIZE-SIMPLEX(A, b, c)
let Δ be a new vector of length n
while some index j ∈ N has c_j > 0
    choose an index e ∈ N for which c_e > 0
    for each index i ∈ B
        if a_ie > 0
            Δ_i = b_i / a_ie
        else Δ_i = ∞
    choose an index l ∈ B that minimizes Δ_i
    if Δ_l == ∞
        return "unbounded"
    else (N, B, A, b, c, ν) = PIVOT(N, B, A, b, c, ν, l, e)
for i = 1 to n
    if i ∈ B
        x̄_i = b_i
    else x̄_i = 0
return (x̄_1, x̄_2, ..., x̄_n)
```

PIVOT$(N, B, A, b, c, \nu, l, e)$

```
1  // Compute the coefficients of the equation for new basic variable x_e.
2  let Â be a new m × n matrix
3  b̂_e = b_l / a_le
4  for each j ∈ N − {e}
5      â_ej = a_lj / a_le
6  â_el = 1 / a_le
7  // Compute the coefficients of the remaining constraints.
8  for each i ∈ B − {l}
9      b̂_i = b_i − a_ie b̂_e
10     for each j ∈ N − {e}
11         â_ij = a_ij − a_ie â_ej
12     â_il = −a_ie â_el
13 // Compute the objective function.
14 ν̂ = ν + c_e b̂_e
15 for each j ∈ N − {e}
16     ĉ_j = c_j − c_e â_ej
17 ĉ_l = −c_e â_el
18 // Compute new sets of basic and nonbasic variables.
19 N̂ = N − {e} ∪ {l}
20 B̂ = B − {l} ∪ {e}
21 return (N̂, B̂, Â, b̂, ĉ, ν̂)
```

对于剩下的下标 $i \in B - \{l\}$，我们有

$$\hat{b}_i = b_i - a_{ie}\hat{b}_e \quad \text{（根据 PIVOT 的第 9 行）}$$
$$= b_i - a_{ie}(b_l / a_{le}) \quad \text{（根据 PIVOT 的第 3 行）}$$

$$b_l / a_{le} \leqslant b_i / a_{ie}$$

$$\hat{b}_i = b_i - a_{ie}(b_l / a_{le}) \quad \text{（根据式(29.76)）}$$
$$\geqslant b_i - a_{ie}(b_i / a_{ie}) \quad \text{（根据式(29.77)）}$$
$$= b_i - b_i = 0$$

Maintenance:

第一部分:

第二部分:

第三部分：基本解显然是该线性规划的一个可行解。

Termination:

两种结束方式:

在第3行中终止，则所有的循环不变式都满足。当前线性规划的基本解是可行的。

若在11行中终止（unbounded）,则$a_{ie} \le 0$.

定义:

$$\bar{x}_i = \begin{cases} \infty & 若\ i = e \\ 0 & 若\ i \in N - \{e\} \\ b_i - \sum_{j \in N} a_{ij}\, \bar{x}_j & 若\ i \in B \end{cases}$$

$$\bar{x}_i = b_i - \sum_{j \in N} a_{ij}\, \bar{x}_j = b_i - a_{ie}\, \bar{x}_e$$

$$z = v + \sum_{j \in N} c_j\, \bar{x}_j = v + c_e \bar{x}_e$$

```
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13  for i = 1 to n
14      if i ∈ B
15          x̄_i = b_i
16      else x̄_i = 0
17  return (x̄_1, x̄_2, ..., x̄_n)
```

找到一个可行解，目标值为正无穷，因此原线性规划是无界的。