习题2-4

CS第1.1节问题9、13

CS第1.2节问题**15**

CS第1.3节问题6、9、14

CS第1.5节问题**8**、**10**、**15**

1.2-15 A tennis club has $2n$ members. We want to pair up the members by twos for singles matches. In how many ways can we pair up all the members of the club? Suppose that in addition to specifying who plays whom, we also determine who serves first for each pairing. Now in how many ways can we specify our pairs?

首先，我们在$2n$个人中选出2人作为第一对选手，再从$2n-2$个人中选出2人作为第二对选手，以此类推，一共有

$$\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{4}{2}\binom{2}{2}$$

种选法， 然而这里我们考虑了各队选手的次序， 所以要除以$n!$因此答案为

$$\frac{\binom{2n}{2}\binom{2n-2}{2}\binom{2n-4}{2}\cdots\binom{4}{2}\binom{2}{2}}{n!}=\frac{\binom{2n}{2,2,\cdots,2}}{n!}$$

如果考虑对手的先后顺序，只要把前面的组合改为排列即可，答案为

$$\begin{aligned}&\frac{\mathbf{P}_{2n}^{2}\mathbf{P}_{2n-2}^{2}\mathbf{P}_{2n-4}^{2}\cdots\mathbf{P}_{4}^{2}\mathbf{P}_{2}^{2}}{n!}\\&=\frac{2n!}{n!}\\&=\mathbf{P}_{2n}^{n}\end{aligned}$$

1.5-8 The formula for the number of multisets is $(n + k - 1)!$ divided by a product of two other factorials. We want to use the quotient principle to explain why this formula counts multisets. The formula for the number of multisets is also a binomial coefficient, so it should have an interpretation that involves choosing k items from $n + k - 1$ items. The parts of the problem that follow lead us to these explanations.

a. In how many ways can you place k red checkers and $n - 1$ black checkers in a row?

b. How can you relate the number of ways of placing k red checkers and $n - 1$ black checkers in a row to the number of k-element multisets of an n-element set (the set $\{1, 2, \ldots, n\}$ to be specific)?

c. How can you relate the choice of k items out of $n + k - 1$ items to the placement of red and black checkers, as in parts a and b? Think about how this relates to placing k identical books and $n - k$ identical blocks of wood in a row.

1.5-10 How many solutions to the equation $x_1 + x_2 + \cdots + x_n = k$ are there with each x_i a positive integer?

$\binom{n+k-1}{k}$ **?** n类物品抽取k个可重复？

$$(x_1 - 1) + (x_2 - 1) + \cdots (x_n - 1) = k - n$$
$$x_i > 0$$

令$y_i = x_i - 1$

$$y_1 + y_2 + \cdots y_n = k - n$$
$$y_i \geq 0$$

n类物品抽取k-n个可重复？

$$\binom{k-n+n-1}{k-n} = \binom{k-1}{k-n}$$

The following table gives the solutions to the counting problems.

Elements of N	Elements of M	Any f	Injective (1-1) f	Surjective (on-to) f
distinguishable	*distinguishable*	m^n	$(m)_n$	$m!\left\{ n \atop m \right\}$
indistinguishable	*distinguishable*	$\left(\!\binom{m}{n}\!\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
distinguishable	*indistinguishable*	$\sum_{k=1}^{m} \left\{ n \atop k \right\}$	$\begin{cases} 1 & \text{if } n \le m \\ 0 & \text{if } n > m \end{cases}$	$\left\{ n \atop m \right\}$
indistinguishable	*indistinguishable*	$\sum_{k=1}^{m} p_k(n)$	$\begin{cases} 1 & \text{if } n \le m \\ 0 & \text{if } n > m \end{cases}$	$p_m(n)$

$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$ is usually denoted as $(n)_k$, read "n lower factorial k".

$$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$$

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indistinguishable	*indistinguishable*	$\sum_{k=1}^{m} p_k(n)$	$\begin{cases} 1 & \text{if } n \le m \\ 0 & \text{if } n > m \end{cases}$	$p_m(n)$

The number $\left\{ {n \atop k} \right\}$ is called a **Stirling number of the second kind.** $\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$

We define $p_k(n)$ as the number of k-partitions of n

A ***k*-partition** of a number n is a multiset $\{x_1, x_2, \ldots, x_k\}$ with $x_i \ge 1$ for every element x_i and $x_1 + x_2 + \cdots + x_k = n$

$$p(n) = \sum_{k=1}^{n} p_k(n)$$

be the total number of partitions of n. $p(n)$ is called the **partition number**.