作业1-12

SM第14章问题32,44,46,58,62,66,70,75

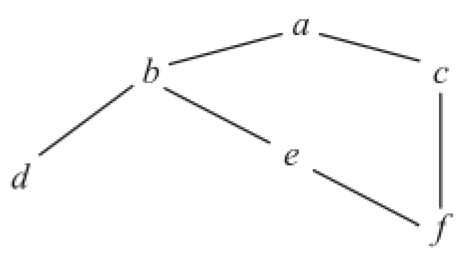
14.32. Let $B = \{a, b, c, d, e, f\}$ be ordered as in Fig. 14-17(b).

(a) Find all minimal and maximal elements of B.

(b) Does B have a first or last element?

(c) List two and find the number of consistent enumerations of B into the set $\{1, 2, 3, 4, 5, 6\}$.

(c)11种



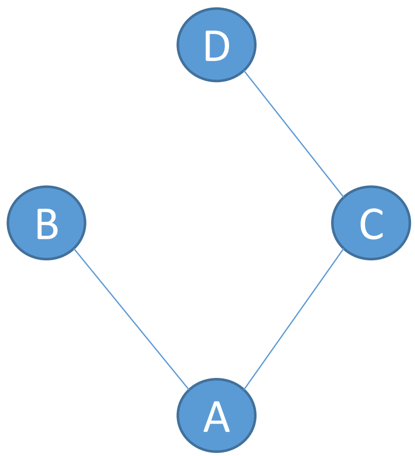
(b)

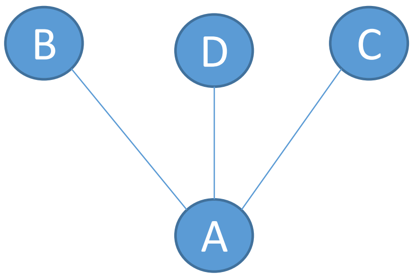
Fig. 14-17

14.44. Suppose the following are three consistent enumerations of an ordered set $A = \{a, b, c, d\}$:

$$[(a, 1), (b, 2), (c, 3), (d, 4)], \quad [(a, 1), (b, 3), (c, 2), (d, 4)], \quad [(a, 1), (b, 4), (c, 2), (d, 3)]$$

Assuming the Hasse diagram D of A is connected, draw D.





14.58. Show that the isomorphism relation $A \cong B$ for ordered sets is an equivalence relation, that is:

(a) $A \cong A$ for any ordered set A. (b) If $A \cong B$, then $B \cong A$. (c) If $A \cong B$ and $B \cong C$, then $A \cong C$.

- Key point:

Two ordered sets X and Y are said to be *isomorphic* or *similar*, written

$$X \simeq Y$$

if there exists a one-to-one correspondence (bijective mapping) $f: X \to Y$ which preserves the order relations, i.e., which is a similarity mapping. **(1)** **(2)**

Suppose X and Y are partially ordered sets. A one-to-one (injective) function $f: X \to Y$ is called a *similarity mapping* from X into Y if f preserves the order relation, that is, if the following two conditions hold for any pair a and a' in X:

(1) If $a \precsim a'$ then $f(a) \precsim f(a')$.

(2) If $a \parallel a'$ (noncomparable), then $f(a) \parallel f(a')$.

14.62. Suppose A and B are well-ordered isomorphic sets. Show that there is only one similarity mapping $f: A \to B$.

- Proof-1

$\because A, B$ are well $-$ ordered

$case(1)$: A, B are finite

A can be denoted as:$\{a_0, a_1, \dots, a_n\}$, where $|A| = n, a_i \precsim a_j$ for $0 \leqq i \leqq j \leqq n$

B can be denoted as:$\{b_0, b_1, \dots, b_n\}$, where $|B| = n, b_i \precsim b_j$ for $0 \leqq i \leqq j \leqq n$

$case(2)$: A, B are infinite

A can be denoted as:$\{a_0, a_1, \dots, a_k, \dots\}$, where $a_i \precsim a_j$ for $0 \leqq i \leqq j$

B can be denoted as:$\{b_0, b_1, \dots, b_k, \dots\}$, where $b_i \precsim b_j$ for $0 \leqq i \leqq j$

For both cases:

$\because$ A,B are isomorphic

$\therefore$ there is a bijective similarity mapping $f: A \to B$:

"For each $a_i \in A$, $f(a_i) = b_i$" by introduction on *i.*

Base: i=0

it is easy to show that $f(a_0) = b_0$

H: for $i \leq k, f(a_i) = b_i$

I: for $i = k + 1$, if $f(a_{k+1}) \neq b_{k+1}$,

$\because f$ is $1 - to - 1$ and for $i \leq k, f(a_i) = b_i$

$\therefore \exists b_j \in B, j > k + 1$ $s.t.$ $f(a_{k+1}) = b_j$

又 $\because f$ is bijective, and for $i \leq k, f(a_i) = b_i$

$\therefore f^{-1}(b_{k+1}) \in A$ and $a_{k+1} \preccurlyeq f^{-1}(b_{k+1})$

$\therefore f(a_{k+1}) = \boldsymbol{b_j} \preccurlyeq f\left(f^{-1}(b_{k+1})\right) = \boldsymbol{b_{k+1}}$, **which is contractive to $\boldsymbol{b_{k+1} \prec b_j (as\ k + 1 < j)}$**

so, the assumption $f(a_{k+1}) \neq b_{k+1}$ **is wrong!** $f(a_{k+1}) = b_{k+1}$

(*) Every element $a \in S$, other than a last element, has an immediate successor. For, let $M(a)$ denote the set of elements which strictly succeed a. Then the first element of $M(a)$ is the immediate successor of a.

Proof-2

$\because A, B$ are well $-$ ordered

$case(1)$: A, B are finite

A can be denoted as:$\{a_0, a_1, \dots, a_n\}$, where $|A| = n, a_i \precsim a_j$ for $0 \leqq i \leqq j \leqq n$

B can be denoted as:$\{b_0, b_1, \dots, b_n\}$, where $|B| = n, b_i \precsim b_j$ for $0 \leqq i \leqq j \leqq n$

$case(2)$: A, B are infinite

A can be denoted as:$\{a_0, a_1, \dots, a_k, \dots\}$, where $a_i \precsim a_j$ for $0 \leqq i \leqq j$

B can be denoted as:$\{b_0, b_1, \dots, b_k, \dots\}$, where $b_i \precsim b_j$ for $0 \leqq i \leqq j$

For both cases:

$\because$ A,B are isomorphic

$\therefore$ there is a bijective similarity mapping $f: A \to B$:

"For each $a_i \in A$, $f(a_i) = b_i$" by introduction on *i.*

Base: i=0

it is easy to show that $f(a_0) = b_0$

H: for $i \leq k, f(a_i) = b_i$

I: for $i = k + 1$, let $f(a_{i+1}) = b_{\mathrm{m}} (m \geq i + 1)$

according to the fact (*), a_{k+1} is the first element of $M(a_k)$, so is b_{k+1}

$\because f$ is bijective, and for $i \leq k, f(a_i) = b_i$

$\therefore f^{-1}(b_{k+1}) \in M(a_k)$ and $a_{k+1} \preccurlyeq f^{-1}(b_{k+1})$

$\therefore f(a_{k+1}) = \boldsymbol{b_m} \preccurlyeq f\left(f^{-1}(b_{k+1})\right) = \boldsymbol{b_{k+1}}$

又 $\because \boldsymbol{b_{k+1}} \preccurlyeq \boldsymbol{b_m}$**(**$b_{k+1}$ is the first element of $M(b_k)$**)**

So, $f(a_{k+1}) = b_{k+1}$

14.62. Suppose A and B are well-ordered isomorphic sets. Show that there is only one similarity mapping $f: A \to B$.

- Assume there are two different bijective similarity mapping $f: A \to B$, and $g: A \to B$
- As *f, g are different,* define $C = \{a | a \in A, f(a) \neq g(a)\}$. Obviously ,$C \neq \emptyset$
- $\because C \subseteq A$ and A is well-ordered

  $\therefore C$ is well-ordered, and let $c \in$ C be the first element of C;
- It is easy to show that for every $a' \in \mathrm{A}, a' \prec c$, we have $f(a') = g(a')$
- Let $f(c) = x, g(c) = y, x \neq y$, then x and y are comparable(why?), without losing generality, assume $x \preccurlyeq y$
  - As g is bijective, then $g^{-1}(x) \in A - S_A(c)$, here $S_A(c) = \{p | p \in A, p \prec c\}$, i.e., $g^{-1}(x) \in \{p | p \in A, c \preccurlyeq p\}$
  - $\therefore g(c) \preccurlyeq g\left(g^{-1}(x)\right)$
  - $\therefore y \preccurlyeq x$
  - $\therefore x = y, f(c) = g(c)$, contradicting to C's definition
- So, f and g should be the same.

14.66. Consider the lattice M in Fig. 14-19(b).

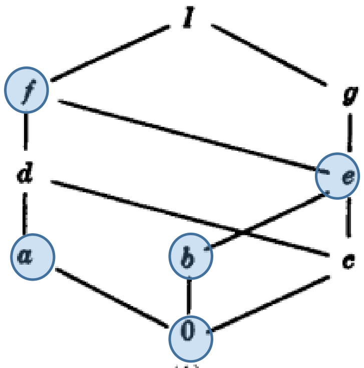
(a) Find all join-irreducible elements.

(b) Find the atoms.

(c) Find complements of a and b, if they exist.

(d) Express each x in M as the join of irredundant join-irreducible elements.

(e) Is M distributive? Complemented?



(b)

- Does M contain a sub-lattice isomorphic to

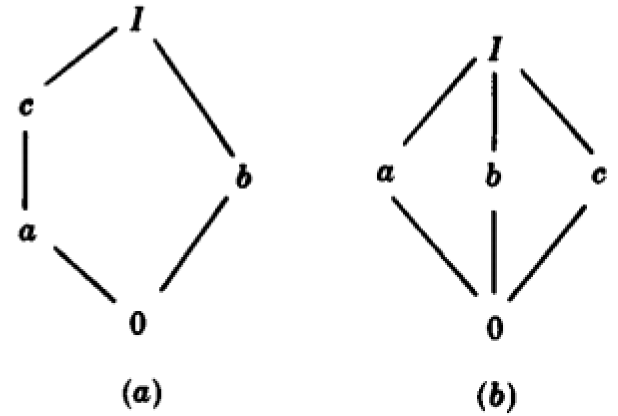


Fig. 14-7

Suppose M is a nonempty subset of a lattice L. We say M is *a sub-lattice* of L if M itself is a lattice (**with respect to the operations of** L).

We note that M is a sub-lattice of L if and only if M is **closed under the operations of** $\wedge$ **and** $\vee$ **of** L.

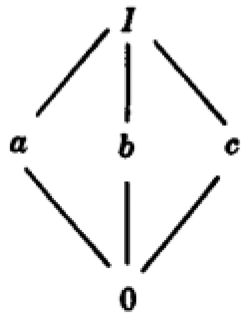
14.75. A lattice M is said to be *modular* if whenever $a \le c$ we have the law

$$a \vee (b \wedge c) = (a \vee b) \wedge c$$

(a) Prove that every distributive lattice is modular.

(b) Verify that the non-distributive lattice in Fig. 14-7(b) is modular; hence the converse of (a) is not true.

(c) Show that the nondistributive lattice in Fig. 14-7(a) is non-modular. (In fact, one can prove that every non-modular lattice contains a sublattice isomorphic to Fig. 14-7(a).)



(b)

$$x \vee (y \wedge z) = (x \vee y) \wedge z$$

Discussion on different cases on possible values of x,y,z:

Case 1: $x = 0$

$$x \vee (y \wedge z) = 0 \vee (y \wedge z) = y \wedge z$$
$$(x \vee y) \wedge z = (0 \vee y) \wedge z = y \wedge z$$

Case 2: $x \in \{a, b, c\}$

- **Case 2.1:** $z = x$

$$x \vee (y \wedge z) = x \vee (y \wedge x) = x$$
$$(x \vee y) \wedge z = (x \vee y) \wedge x = x$$

- **Case 2.2:** $z = I$

$$x \vee (y \wedge z) = x \vee (y \wedge I) = x \vee y$$
$$(x \vee y) \wedge z = (x \vee y) \wedge I = x \vee y$$

Case 3: $x = I, z = I$

$$x \vee (y \wedge z) = I \vee (y \wedge I) = I$$
$$(x \vee y) \wedge z = (I \vee y) \wedge I = I$$