作业1-9

UD第10章问题2、4、5、8

UD第11章问题**3**、**7**、8、9

UD第12章问题**10**、13b、16、20、22、23

UD第27章项目4

Problem 11.3. (a) For each $r \in \mathbb{R}$, let $A_r = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = r\}$. Is this a partition of $\mathbb{R}^3$? If so, give a geometric description of the partitioning sets.

(b) For each $r \in \mathbb{R}$, let $A_r = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = r^2\}$. Is this a partition of $\mathbb{R}^3$? If so, give a geometric description of the partitioning sets.

nition. We turn to that now. A **partition of a nonempty set** X is a collection $\mathscr{A}$ of subsets of X that satisfies the following three conditions.

(i) Every set $A \in \mathscr{A}$ is nonempty,
(ii) $\bigcup_{A\in\mathscr{A}} A = X$, and
(iii) for all $A, B \in \mathscr{A}$, if $A \cap B \neq \emptyset$, then $A = B$.

a) Yes,

I. $\forall r \in \mathbb{R}$, we have $(0,0,r) \in A_r$, so $A_r \neq \emptyset$

II. Try to show $\bigcup_{r\in\mathbb{R}} A_r = \mathbb{R}^3$

① First, it is obvious that $\forall r \in \mathbb{R}, A_r \subseteq \mathbb{R}^3$, so $\bigcup_{r\in\mathbb{R}} A_r \subseteq \mathbb{R}^3$

② Second, for each $(a,b,c) \in \mathbb{R}^3$, we have $a+b+c = r_0 \in \mathbb{R}$, so $(a,b,c) \in A_{r_0}$; as a result, $(a,b,c) \in \bigcup_{r\in\mathbb{R}} A_r$ And consequently, $\mathbb{R}^3 \subseteq \bigcup_{r\in\mathbb{R}} A_r$

③ Therefore, $\bigcup_{r\in\mathbb{R}} A_r = \mathbb{R}^3$

III. $\forall r_1, r_2 \in \mathbb{R}$, if $A_{r_1} \cap A_{r_2} \neq \emptyset$, then $A_{r_1} = A_{r_2}$

$\because A_{r_1} \cap A_{r_2} \neq \emptyset$

$\therefore \exists x \in A_{r_1} \cap A_{r_2}$

assume x=(a,b,c), we have $a+b+c = r_1$ and $a+b+c = r_2$

$\therefore r_1 = r_2$

$\therefore A_{r_1} = A_{r_2}$

Problem 11.3. (a) For each $r \in \mathbb{R}$, let $A_r = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = r\}$. Is this a partition of $\mathbb{R}^3$? If so, give a geometric description of the partitioning sets.

(b) For each $r \in \mathbb{R}$, let $A_r = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = r^2\}$. Is this a partition of $\mathbb{R}^3$? If so, give a geometric description of the partitioning sets.

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b) Yes,

$A_1 = A_{-1}$?

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III. $\forall r_1, r_2 \in \mathbb{R}$, if $A_{r_1} \cap A_{r_2} \neq \emptyset$, then $A_{r_1} = A_{r_2}$

$\because A_{r_1} \cap A_{r_2} \neq \emptyset$

$\therefore \exists x \in A_{r_1} \cap A_{r_2}$

assume x=(a,b,c), we have$a^2+b^2+c^2 = r_1^2$ and $a^2+b^2+c^2 = r_2^2$

$\therefore r_1 = r_2$

$\therefore A_{r_1} = A_{r_2}$

Problem 11.7. Consider the set P of polynomials with real coefficients. Decide whether or not each of the following collection of sets determines a partition of P. If you decide that it does determine a partition, show it carefully. If you decide that it does not determine a partition, list the part(s) of the definition that is (are) not satisfied and justify your claim with an example. (See Problem 10.8 for more information about polynomials.)

(a) For $m \in \mathbb{N}$, let A_m denote the set of polynomials of degree m. The collection of sets is $\{A_m : m \in \mathbb{N}\}$.

(b) For $c \in \mathbb{R}$, let A_c denote the set of polynomials p such that $p(0) = c$. The collection of sets is $\{A_c : c \in \mathbb{R}\}$.

(c) For a polynomial q, let A_q denote the set of all polynomials p such that q is a factor of p; that is, there is a polynomial r such that $p = qr$. The collection of sets is $\{A_q : q \in P\}$.

(d) For $c \in \mathbb{R}$, let A_c denote the set of polynomials p such that $p(c) = 0$. The collection of sets is $\{A_c : c \in \mathbb{R}\}$.

Problem 10.8. Recall that a **polynomial** p over $\mathbb{R}$ is an expression of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$ where each $a_j \in \mathbb{R}$ and $n \in \mathbb{N}$. The largest integer j such that $a_j \neq 0$ is the **degree** of p. We define the degree of the constant polynomial $p = 0$ to be $-\infty$. (A polynomial over $\mathbb{R}$ defines a function $p : \mathbb{R} \to \mathbb{R}$.)

(a) For $m \in \mathbb{N}$, let A_m denote the set of polynomials of degree m. The collection of sets is $\{A_m : m \in \mathbb{N}\}$.

I. $\forall m \in \mathbb{N}$, it is obvious that $x^m \in A_m$, so $A_m \neq \emptyset$

II. To show $\bigcup_{m\in\mathbb{N}} A_m = P$

① It is obvious that $A_m \subseteq$ P, so $\bigcup_{m\in\mathbb{N}} A_m \subseteq P$

② For each $p \in P$, we have $p \in A_{(\deg(p))} \subseteq \bigcup_{m\in\mathbb{N}} A_m$. Therefore, $P \subseteq \bigcup_{m\in\mathbb{N}} A_m$

③ So, $\bigcup_{m\in\mathbb{N}} A_m = P$

III. To show $\forall m_1, m_2 \in \mathbb{N}$, if $A_{m_1} \cap A_{m_2} \neq \emptyset$, then $A_{m_1} = A_{m_2}$

① $\because A_{m_1} \cap A_{m_2} \neq \emptyset$

② $\therefore \exists p \in A_{m_1} \cap A_{m_2}$, so $\deg(p) = m_1 \; and \; \deg(p) = m_2$

③ $\therefore m_1 = m_2$, so $A_{m_1} = A_{m_2}$

Special case: $\boldsymbol{p(x) = 0}$

$\mathbf{deg}(\boldsymbol{p}) = -\infty$

(c) For a polynomial q, let A_q denote the set of all polynomials p such that q is a factor of p; that is, there is a polynomial r such that $p = qr$. The collection of sets is $\{A_q : q \in P\}$.

$p = (x+1)(x+2)$
$q = (x+1)$
$r = (x+2)$

1. $p \in A_q$
2. $r \in A_q$

?

Ans: No

$p = (x+1)(x+2)$
$q = (x+1)$
$r = (x+2)$

So, $p = qr = rq$

$\therefore p \in A_q$ and $p \in A_r$

But, $A_q \neq A_r$

Problem 12.10. Let S and T be nonempty bounded subsets of $\mathbb{R}$.

(a) Show that $\sup(S \cup T) \geq \sup S$, and $\sup(S \cup T) \geq \sup T$.

(b) Show that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

(c) Try to state the results of (a) and (b) in English, without using mathematical symbols.

(a)设 A=sup S, B=sup T

∀a∈S , a≤A

∀b∈T , b≤B

不妨设 A≥B

b≤B≤A

所以(S∪T)中任意一个元素≤A,

又因为 A 是 S 中的元素，所以 A 也是(S∪T)中的元素

⇒ A= sup(S∪T)

⇒ sup(S∪T) ≥sup S, sup(S∪T)≥sup T

Problem 12.10. Let S and T be nonempty bounded subsets of $\mathbb{R}$.

(a) Show that $\sup(S \cup T) \geq \sup S$, and $\sup(S \cup T) \geq \sup T$.
(b) Show that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.
(c) Try to state the results of (a) and (b) in English, without using mathematical symbols.

Let $\mathrm{A} = \sup S, B = \sup T, C = \sup(S \cup T)$

(a)
Obviously， $\forall x, x \in (S \cup T) \Rightarrow x < C$
$\therefore \forall x, x \in S \Rightarrow x < C; \forall x, x \in T \Rightarrow x < C$
$\therefore$C is a upper bound of both S and T
As $\mathrm{A} = \sup S, B = \sup T$, by the definition of supremum, we have:
$A \leq C$ and $B \leq C$

(b)From (a) we got $\boldsymbol{C \geq max\{A, B\}}$, we only need to prove $\boldsymbol{C \leq max\{A, B\}}$,
Without losing generality, assume $A \geq B$, then:

- $\forall x \in S, x \leq A$
- $\forall x \in T, x \leq B \leq A$

$\therefore \forall x, x \in (S \cup T) \Rightarrow x \leq A$
$\therefore$ A is a upper bound of $S \cup T$
As $C = \sup(S \cup T)$, by the definition of supremum, we have:
$C = \sup(S \cup T) \leq A$
Consequently, $\boldsymbol{C = max\{A, B\}}$

Problem(12.16(e))
Prove that $(\mathcal{P}(\mathbb{Z}), \subseteq)$ has the ”least upper set property”(in other words, show every upper bounded set has a least upper set)

- Idea: construct and prove
  - Construct: for $\mathcal{A} \subseteq \mathcal{P}(\mathbb{Z})$, we can obtain a set *C* by:

$$C = \bigcup_{X_i \in \mathcal{A}} X_i$$

  - Prove: try to show C is the least upper set
    - Assume *U* is the least upper set of $\mathcal{A}$
    - $\because \forall X_i \in \mathcal{A},$ it’s obvious that $X_i \subseteq \bigcup_{X_i \in \mathcal{A}} X_i = C$
    - $\therefore C$ is an upper set of $\mathcal{A}$, i.e., $U \subseteq C$
    - Then, $\forall X_i \in \mathcal{A} \Rightarrow X_i \subseteq U$
    - $\therefore C = \bigcup_{X_i \in \mathcal{A}} X_i \subseteq U$
    - $\therefore C = U$

Problem(12.23)

Prove that for two arbitrary real numbers a and b with $a < b$, there is an irrational number c such that $a < c < b$.

(Hint: Consider $\frac{a}{\sqrt{2}}$ and $\frac{b}{\sqrt{2}}$)

- $\frac{a}{\sqrt{2}}$ and $\frac{b}{\sqrt{2}}$ are real numbers, *a*<*b*
- By Theorem 12.11, there is a rational number c' such that: $\frac{a}{\sqrt{2}} < c' < \frac{b}{\sqrt{2}}$
- $\therefore a < \sqrt{2}c' < b$, let $c = \sqrt{2}c'$
- Now, we have to show c is an irrational
  - H: assume c is a rational number, then $\exists p, q \in \mathbb{Z}, q \neq 0$,s.t. $c = \frac{p}{q}$
  - As c' is also a rational number, then $\exists p', q' \in \mathbb{Z}, q' \neq 0$,s.t. $c' = \frac{p'}{q'}$
  - So, $c = \frac{p}{q} = \sqrt{2}c' = \frac{\sqrt{2}p'}{q'}$
  - $\therefore \sqrt{2} = \frac{pq'}{p'q}$ should be an rational number, contracting with the fact that $\sqrt{2}$ is irrational.
  - $\therefore$ H is not right, and c is an irrational