## Rotational Symmetries

Hengfeng Wei<br>hfwei@nju.edu.cn<br>March 19 ~ March 23, 2017<br>

## Rotational Symmetries

(1) Rotational Symmetries of Tetrahedron
(2) Rotational Symmetries of Cube

## Rotational Symmetries

(1) Rotational Symmetries of Tetrahedron
(2) Rotational Symmetries of Cube

## $C \cong S_{4}$

## 6 faces/8 vertices/12 edges

## $C \cong S_{4}$

# 6 faces/8 vertices/12 edges 

$$
|C| \leq 24
$$

## $C \cong S_{4}$

# 6 faces/8 vertices/12 edges 

$$
|C| \leq 24
$$

1. facing upward
2. 24 oriented edges

## $C \cong S_{4}$

# 6 faces/8 vertices/12 edges 

$$
|C| \leq 24
$$

1. facing upward
2. 24 oriented edges
$|C|=24 \Leftarrow 4$ main diagonals

## $C \cong S_{4}$

- Order of 1: id $(\#=1)$
- Order of 4: face-to-face

$$
\begin{aligned}
& f_{t d}=\left(\begin{array}{ll}
1 & 2
\end{array} 34\right) \quad f_{t d}^{2}=(13)(24) \quad f_{t d}^{3}=\left(\begin{array}{lll}
1 & 4 & 3
\end{array}\right) \\
& f_{l r}=\left(\begin{array}{ll}
1 & 2
\end{array} \text { 3) } \quad f_{l r}^{2}=(14)(23) \quad f_{l r}^{3}=(1342)\right. \\
& f_{f b}=(1423) \quad f_{f b}^{2}=(12)(34) \quad f_{f b}^{3}=\left(\begin{array}{ll}
1 & 3
\end{array}\right)
\end{aligned}
$$

## $C \cong S_{4}$

- Order of 3: vertex-to-vertex

$$
\begin{aligned}
& v_{1}=(234) \quad v_{1}^{2}=\left(\begin{array}{ll}
2 & 4
\end{array}\right) \\
& v_{2}=\left(\begin{array}{ll}
1 & 4
\end{array}\right) \quad v_{2}^{2}=\left(\begin{array}{lll}
1 & 3 & 4
\end{array}\right) \\
& v_{3}=\left(\begin{array}{lll}
1 & 2
\end{array}\right) \quad v_{3}^{2}=\left(\begin{array}{ll}
1 & 4
\end{array}\right) \\
& v_{4}=\left(\begin{array}{lll}
1 & 2 & 3
\end{array}\right) \quad v_{4}^{2}=\left(\begin{array}{lll}
1 & 3 & 2
\end{array}\right)
\end{aligned}
$$

- Order of 2: edge-to-edge

$$
\begin{array}{lll}
e_{12}=\left(\begin{array}{ll}
1 & 2
\end{array}\right) & e_{13}=\left(\begin{array}{ll}
1 & 3
\end{array}\right) & e_{14}=\left(\begin{array}{ll}
1 & 4
\end{array}\right) \\
e_{23}=\left(\begin{array}{ll}
2 & 3
\end{array}\right) & e_{24}=\left(\begin{array}{ll}
2 & 4
\end{array}\right) & e_{34}=\left(\begin{array}{ll}
3 & 4
\end{array}\right)
\end{array}
$$

## Subgroups of $S_{4}$

Possible orders: $\begin{array}{llllllll}1 & 2 & 3 & 4 & 6 & 8 & 12 & 24\end{array}$

- $|H|=1: \#=1$
- $|H|=24: \#=1$
- $|H|=2: \#=6+3=9$
- $|H|=3: \#=4$


## Subgroups of order 4

- $H \cong \mathbb{Z}_{4}: \#=3$
- $H \cong K_{4}=\{e, a, b, c\}\left(a^{2}=b^{2}=c^{2}\right)$

$$
\begin{gathered}
\left\{(1),(12),\left(\begin{array}{ll}
3 & 4
\end{array}\right),\left(\begin{array}{ll}
1 & 2
\end{array}\right)\left(\begin{array}{l}
3
\end{array}\right)\right\} \\
\{(1),(13),(24),(13)(24)\} \\
\{(1),(14),(23),(14)(23)\} \\
\left.\{(1),(12)(13),(2) 4),\left(\begin{array}{ll}
1 & 4
\end{array}\right)(23)\right\} \\
\#=3+4=7
\end{gathered}
$$

## Subgroups of order 6

$$
\begin{gathered}
H \nsupseteq \mathbb{Z}_{6} \\
H \cong S_{3}=\left\{1, r, r^{2}, s, r s, r^{2} s\right\} \quad\left(r^{3}=1, s^{2}=1, s r s=r^{-1}\right)
\end{gathered}
$$

Figure here.
Theorem
There are only 4 subgroups of order 6 in $S_{4}$.

$$
\begin{gathered}
r=\left(\begin{array}{lll}
1 & 3 & 2
\end{array}\right), \quad s=\left(\begin{array}{ll}
1 & 3
\end{array}\right) \\
\text { What does } s r s=r^{-1} \text { mean? }
\end{gathered}
$$

## Subgroups of order 8

$$
\begin{gathered}
H \not \not \mathbb{Z}_{8} \\
H \nsubseteq \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \\
H \not \not \mathbb{Z}_{4} \times \mathbb{Z}_{2} \\
H \nsupseteq Q_{8}: \Longrightarrow|H| \geq 9 \\
H \cong D_{4}=\left\{1, r, r^{2}, r^{3}, s, r s, r^{2} s, r^{3} s\right\} \quad\left(r^{4}=1, s^{2}=1, s r s=r^{-1}\right)
\end{gathered}
$$

Figure here.
Theorem
There are only 3 subgroups of order 8 of $S_{4}$.

## Subgroups of order 12

$$
\begin{gathered}
H \cong \mathbb{Z}_{12}, \mathbb{Z}_{6} \times \mathbb{Z}_{2}, D_{6}, A_{4}, D i c_{12} \\
H \cong A_{4}
\end{gathered}
$$

Figure here.
Theorem
There is only one subgroup of order 12 in $S_{4}$.
Proof.

