# 2-11 Heapsort

Hengfeng Wei

hfwei@nju.edu.cn

May 12, 2020

# Obama in a job interview at Google

"What is most efficient way to sort a million 32-bit integers?"

Obama: "The bubblesort would be the wrong way to go."

Obama: "The bubblesort would be the wrong way to go."

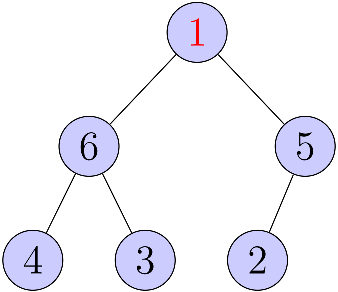
$O \qquad \Omega \qquad \Theta$


Best case　　Worst case　　Average case

## Worst-case of MAX-HEAPIFY (TC 6.2-6)

Show that the worst-case running time of MAX-HEAPIFY on an $n$-element heap is $\Omega(\log n)$.

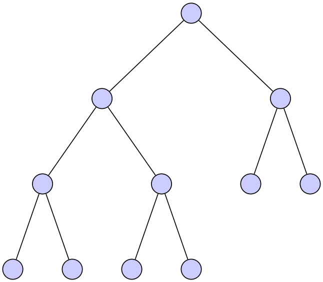
By Example.




COMPARE *vs.* EXCHANGE

## Worst-case of MAX-HEAPIFY (Section 6.2 of CLRS)

Show that the worst-case running time of MAX-HEAPIFY on an $n$-element heap is $O(\log n)$.



$$W(n) \le H(n)$$

No Examples Here!

*Therefore . . .*

**Worst-case of MAX-HEAPIFY**

Show that the worst-case running time of MAX-HEAPIFY on an $n$-element heap is $\Theta(\log n)$.

| | $O$ | $\Omega$ | $\Theta$ |
|---|---|---|---|
| *Worst-case* | "power" of $\mathcal{A}$ | by example | $O = \Omega$ |

## Worst-case of HEAPSORT (TC 6.4-4)

Show that the worst-case running time of HEAPSORT is $\Omega(n \log n)$.

By Example.

Non-proof.

$$\underbrace{\Theta(n)}_{\text{EXTRACT-MAX}} \times \underbrace{\Omega(\log n)}_{\text{MAX-HEAPIFY}} = \Omega(n \log n)$$

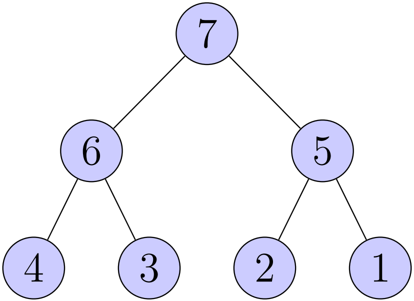
What is wrong?

□

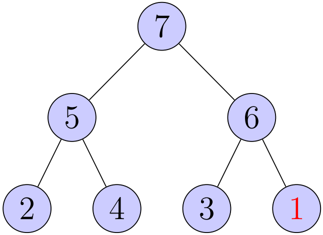
### Worst-case of HEAPSORT (TC $6.4-4$)

Show that the worst-case running time of HEAPSORT is $\Omega(n \log n)$.

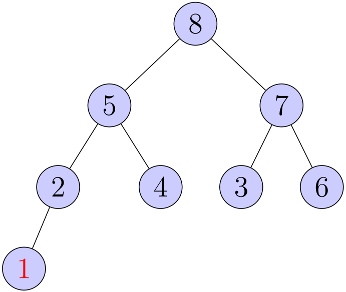
# Heap in decreasing order?




$$T(7) = 2 + 1 + 1 + 1 + 0 + 0 = 5$$




$$T(7) = 2 + 2 + 2 + 1 + 1 + 0 = 8$$




(Ex. 23, Section 5.2.3, TAOCP Vol 3)

$$\sum_{r=1}^{n-1} \lfloor \log r \rfloor = n \lfloor \log n \rfloor - 2^{\lfloor \log n \rfloor + 1} + 2 = \Omega(n \log n)$$

Worst-case of HEAPSORT (TC 6.4 – 4)

Show that the worst-case running time of HEAPSORT is $O(n \log n)$.

$$\sum_{r=1}^{n-1} \lfloor \log r \rfloor = n \lfloor \log n \rfloor - 2^{\lfloor \log n \rfloor + 1} + 2 = O(n \log n)$$

No Examples Here!

$$\underbrace{\Theta(n)}_{\text{EXTRACT-MAX}} \times \underbrace{O(\log n)}_{\text{MAX-HEAPIFY}} = O(n \log n)$$

## Therefore . . .

**Worst-case of HEAPSORT (TC 6.4 – 4)**

Show that the worst-case running time of HEAPSORT is $\Theta(n \log n)$.

| | $O$ | $\Omega$ | $\Theta$ |
|---|---|---|---|
| *Worst-case* | "power" of $\mathcal{A}$ | by example | $O = \Omega$ |

Algorithm $\mathcal{A}$

Inputs $\mathcal{I}$ of size $n$

| | $O$ | $\Omega$ | $\Theta$ |
|---|---|---|---|
| *Best-case* | by example | "weakness" of $\mathcal{A}$ | $O = \Omega$ |
| *Worst-case* | "power" of $\mathcal{A}$ | by example | $O = \Omega$ |

**Best-case of HEAPSORT (TC 6.4-5*)**

Show that when all elements are distinct, the best-case running time of HEAPSORT is $\Omega(n \log n)$.

**Best-case of HEAPSORT (Ex. 32, Section 5.2.3, TAOCP Vol 3)**

Prove that the number of heapsort promotions, $B$, is always at least $\frac{1}{2} N \log N + O(N)$, if the keys being sorted are distinct.

## Best-case of HEAPSORT (TC 6.4-5$^\star$)

Show that when all elements are distinct, the best-case running time of HEAPSORT is $\Omega(n \log n)$.

Consider the largest $m = \lceil n/2 \rceil$ elements.

The largest $m$ elements form a subtree.

$\geq \lfloor m/2 \rfloor$ of $m$ must be nonleaves of that subtree.

$\geq \lfloor m/2 \rfloor$ of $m$ appear in the first $\lfloor n/2 \rfloor$ positions.

They must be promoted to the root before being EXTRACT-MAX.

$$\sum_{k=1}^{\lfloor m/2 \rfloor} \lfloor \log k \rfloor = \frac{1}{2} m \log m + O(m)$$

$$B(n) \geq \frac{1}{4} n \log n + O(n) + B(\lfloor n/2 \rfloor) \implies B(n) \geq \frac{1}{2} n \log n + O(n)$$

Best-case of HEAPSORT (TC 6.4 – 5)

Show that when all elements are distinct, the best-case running time of HEAPSORT is $O(n \log n)$.

By Example.

# “On the Best Case of Heapsort” (1994)

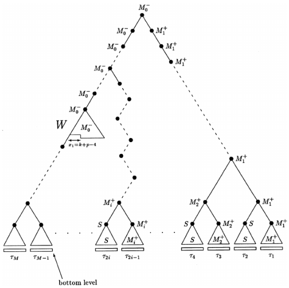



Fig. 2. Initial heap (more detailed).

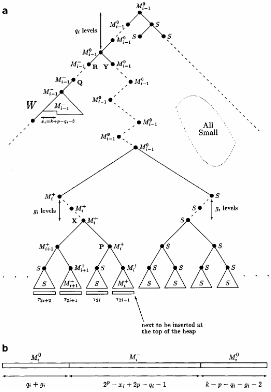



Fig. 3. (a) Odd $i$; (b) contents of the bottom level of $\tau_{2i-1}$, $i$ odd.

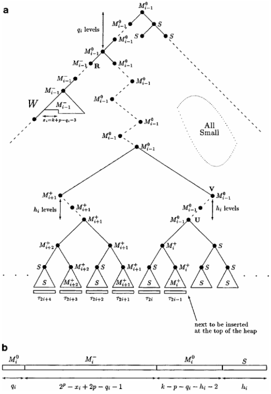



Fig. 4. (a) Even $i$; (b) contents of the bottom level of $\tau_{2i-1}$, $i$ even.

$$B(n) \leq \frac{1}{2} n \log n + O(n \log \log n)$$

Therefore . . .

**Best-case of HEAPSORT (TC 6.4 – 5)**

Show that when all elements are distinct, the best-case running time of HEAPSORT is $\Theta(n \log n)$.

| | $O$ | $\Omega$ | $\Theta$ |
|---|---|---|---|
| *Best-case* | by example | "weakness" of $\mathcal{A}$ | $O = \Omega$ |

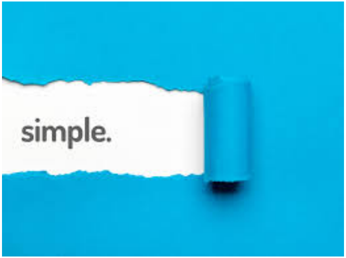
Algorithm $\mathcal{A}$

Inputs $\mathcal{I}$ of size $n$

| | $O$ | $\Omega$ | $\Theta$ |
|---|---|---|---|
| *Best-case* | by example | "weakness" of $\mathcal{A}$ | $O = \Omega$ |
| *Worst-case* | "power" of $\mathcal{A}$ | by example | $O = \Omega$ |
| *Average-case* | $\leq$ | $\geq$ | $O = \Omega$ |

**Average-case of HEAPSORT**

Assume that all elements are distinct. Show that the average-case running time of HEAPSORT is $\Theta(n \log n)$.




I said simple,
not easy.

“By a surprisingly short counting argument.”

Robert Sedgewick


D. E. Knuth

“It is elegant. see exercise 30.”

**Heap Identity (Additional)**

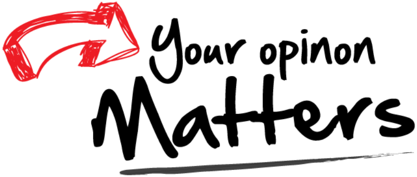
$$\forall h \geq 1 : \lceil \log(\lfloor \frac{1}{2} h \rfloor + 1) \rceil + 1 = \lceil \log(h+1) \rceil$$

$$\boxed{\lceil \log(h+1) \rceil = \lfloor \log h \rfloor + 1, \forall h \geq 1}$$

$$\lfloor \log \lfloor \frac{1}{2} h \rfloor \rfloor + 1 = \lceil \log(\lfloor \frac{1}{2} h \rfloor + 1) \rceil = \lceil \log(h+1) \rceil - 1 = \lfloor \log h \rfloor$$

Depth of $h$ = (Depth of the parent of $h$) + 1


Thank
You!




Office 302

Mailbox: H016

hfwei@nju.edu.cn