1-9 Set Theory (II): Relations

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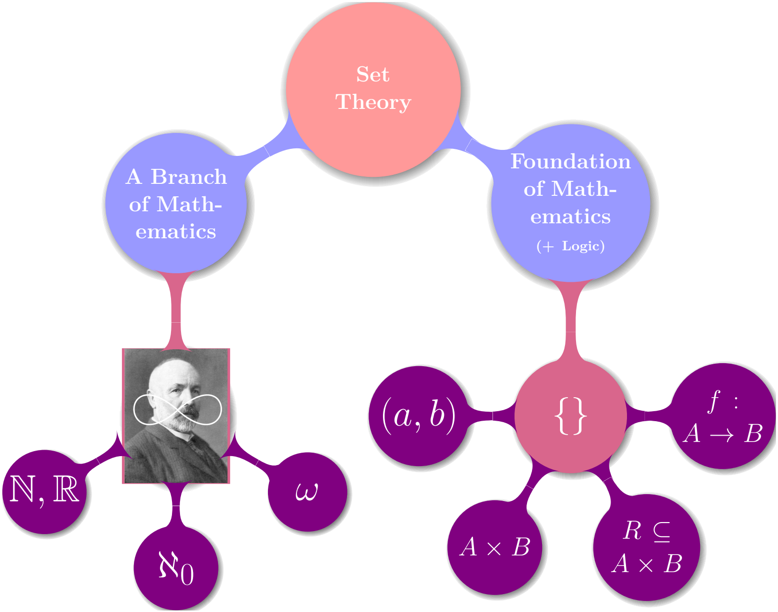
2021 年 12 月 02 日


Set Theory
A Branch of Math-ematics
Foundation of Math-ematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

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$$(a, b) = (c, d) \iff a = c \land b = d$$

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Q : Are you satisfied with the definitions above?

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Definition (Ordered Pairs (Kazimierz Kuratowski; 1921))

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Proof.

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$$

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CASE I : $a = b$

CASE II : $a \neq b$

□

Definition (Ordered Pairs (Norbert Wiener; 1914))

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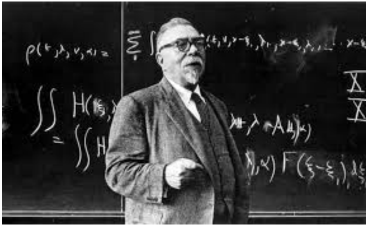

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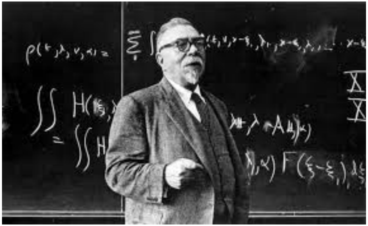

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$A \times B$ *is a set.*

Proof.

$$A \times B \triangleq \{(a, b) \in ? \mid a \in A \wedge b \in B\}$$

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$$\{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$$

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Definition (Notations)

$$(a, b) \in R \qquad R(a, b) \qquad aRb$$

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Examples

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- P : the set of people

$$M = \{(a, b) \in P \times P \mid a \text{ is the mother of } b\}$$

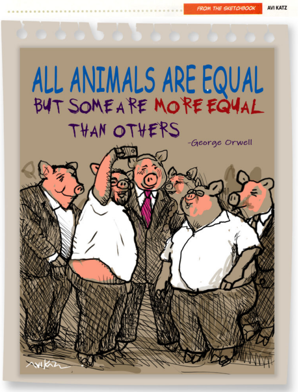
$$B = \{(a, b) \in P \times P \mid a \text{ is the brother of } b\}$$

Important Relations:

Equivalence Relations (1-9)

Functions (1-10)

Ordering Relations (1-12)



3 Definitions

5 Operations

7 Properties

$$R = \{(0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,3)\}$$

3 Definitions

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$$\mathrm{dom}(R) = \{a \mid \exists b : (a, b) \in R\}$$

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$$\{a, b\} \in \bigcup R$$

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Definition (Range)

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Definition (Field)

$$\mathrm{fld}(R) = \mathrm{dom}(R) \cup \mathrm{ran}(R)$$

5 Operations

Definition (Inverse)

The *inverse* of R is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

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Theorem

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Definition (Restriction)

The *restriction* of R to X is the relation

$$R|_X = \{(a, b) \in R \mid a \in X\}$$

Definition (Image)

The *image* of X under R is the set

$$R[X] = \{b \in \mathrm{ran}(R) \mid \exists a \in X : (a, b) \in R\}$$

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Definition (Inverse Image)

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$$R^{-1}[R[X]] \; ? \; X$$

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\Rightarrow & c_0 \in X_1 \land c_0 \notin X_2 : (c_0, b) \in R \\
\iff & \exists a \in X_1 \setminus X_2 : (a, b) \in R
\end{aligned}$$

$$R \subseteq A \times B \qquad X \subseteq A \qquad Y \subseteq B$$

$$R^{-1}[R[X]] \; ? \; X$$

$$R[R^{-1}[Y]] \; ? \; Y$$

Definition (Composition)

The *composition* of relations R and S is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

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$$\leq \circ \leq \ = \ \leq$$

$$\leq \circ \geq \ = \mathbb{R} \times \mathbb{R}$$

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

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$$(a, b) \in (R \circ S)^{-1} \iff \cdots$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

$$(a, b) \in (R \circ S) \circ T \iff \cdots$$

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$$
\begin{aligned}
& (a,b) \in (R \circ S) \circ T \\
\iff & \exists c : (a,c) \in T \land (c,b) \in R \circ S
\end{aligned}
$$

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\begin{aligned}
& (a,b) \in (R \circ S) \circ T \\
\iff & \exists c : (a,c) \in T \land (c,b) \in R \circ S \\
\iff & \exists c : (a,c) \in T \land (\exists d : (c,d) \in S \land (d,b) \in R)
\end{aligned}
$$

$$
\begin{aligned}
& (a,b) \in (R \circ S) \circ T \\
\iff & \exists c : (a,c) \in T \wedge (c,b) \in R \circ S \\
\iff & \exists c : (a,c) \in T \wedge (\exists d : (c,d) \in S \wedge (d,b) \in R) \\
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\end{aligned}
$$

燕小六: “帮我照顾好我七舅姥爷和我外甥女”

“舅姥爷”：姥姥的兄弟

“舅姥爷”: 姥姥的兄弟

$$G = \{(a, b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

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$$G = B \circ (M \circ M)$$

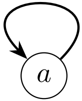
$$G = B \circ (M \circ M) = (B \circ M) \circ M$$

7 Properties

$$R \subseteq X \times X$$

Definition (Reflexive)

$$\forall a \in X : (a, a) \in R$$



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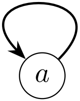

Definition (Irreflexive)

$$\forall a \in X : (a, a) \notin R$$

$$A = \{1, 2, 3\}, R \subseteq A \times A$$

$$\{(1, 1), (1, 3), (2, 1), (2, 2), (3, 3)\}$$

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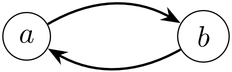
Definition (Symmetric)

$$\forall a, b \in X : aRb \implies bRa$$



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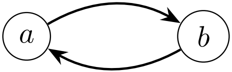

Definition (AntiSymmetric)

$$\forall a, b \in X : (aRb \wedge bRa) \implies a = b$$

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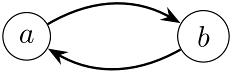

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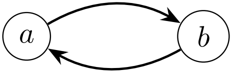

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$>$ *is* antisymmetric.

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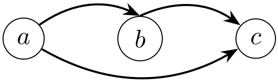
$$R \subseteq X \times X$$

Definition (Transitive)

$$\forall a, b, c \in X : aRb \wedge bRc \implies aRc$$



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$$\{(1, 3)\}$$

$$\emptyset$$

$$R \subseteq X \times X$$

Definition (Connex)

$$\forall a, b \in X : aRb \vee bRa$$

$$R \subseteq X \times X$$

Definition (Connex)

$$\forall a, b \in X : aRb \vee bRa$$

Definition (Trichotomous)

$$\forall a, b \in X : \text{ exactly one of } aRb, bRa, \text{ or } a = b \text{ holds}$$

Theorem

$$R \text{ is reflexive} \iff I \subseteq R$$

$$I = \{(a,a) \in A \times A \mid a \in A\}$$

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Theorem

$$R \text{ is symmetric} \iff R^{-1} = R$$

Theorem

$$R \text{ is transitive} \iff R \circ R \subseteq R$$

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$$(1,2),(2,3),(1,3),(4,4)$$

Equivalence Relations

Definition (Equivalence Relation)

R is an *equivalence relation* on X iff R is

- reflexive
- symmetric
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$$a \sim b \iff a \,\%\, 12 = b \,\%\, 12$$

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Why are equivalence relations important?

Equivalence Relations as Abstractions

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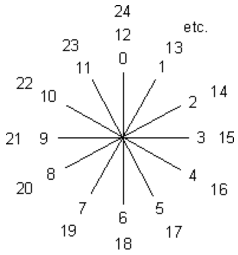
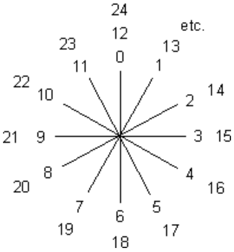
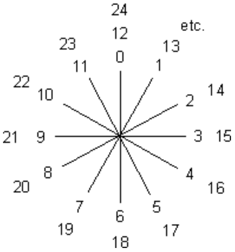


Equivalence Relations as Abstractions



“全国人民代表大会各省代表团”

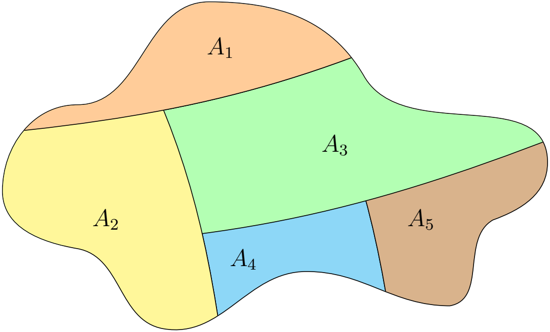
Equivalence Relations as Abstractions



"全国人民代表大会各省代表团"

Equivalence Relation $\Longleftrightarrow$ Partition

Partition



“不空、不漏、不重”

Definition (Partition)

A family of sets $\{A_\alpha : \alpha \in I\}$ is a *partition* of X if

(i)

$$\forall \alpha \in I : A_\alpha \neq \emptyset$$

(ii)

$$\bigcup_{\alpha \in I} A_\alpha = X$$

(iii)

$$\forall \alpha, \beta \in I : A_\alpha \cap A_\beta = \emptyset \vee A_\alpha = A_\beta$$

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Equivalence Relation $R \subseteq X \times X \implies$ Partition Π of X

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The *equivalence class* of *a modulo R* is a set:

$$[a]_R = \{b \in X : aRb\}$$

Equivalence Relation $R \subseteq X \times X \implies$ Partition Π of X

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Definition (Quotient Set)

The *quotient set* is a set:

$$X/R = \{[a]_R \mid a \in X\}$$

Theorem

$$X/R = \{[a]_R \mid a \in X\} \text{ is a partition of } X.$$

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$$\forall x \in X : xRx$$

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$$\forall x, y \in X : xRy \implies yRx$$

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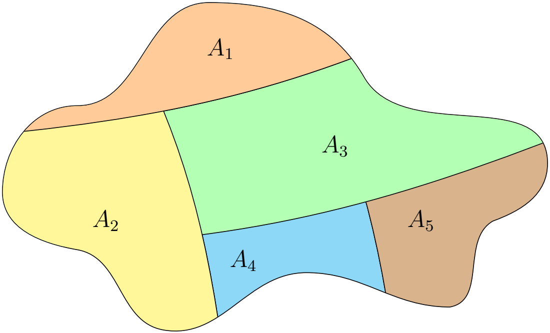
$$\forall x, y, z \in X : xRy \land yRz \implies xRz$$



Equivalence Relation $\Longleftrightarrow$ Partition

Definition

$$\sim \subseteq \mathbb{N}^2 \times \mathbb{N}^2$$

$$(a, b) \sim (c, d) \iff a +_{\mathbb{N}} d = b +_{\mathbb{N}} c$$

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Theorem

$\sim$ *is an equivalence relation.*

Q : What is $\mathbb{N} \times \mathbb{N}/\sim$?

Definition ($\mathbb{Z}$)

$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N}/\sim$$

Definition

$$\sim \subseteq \mathbb{N}^2 \times \mathbb{N}^2$$

$$(a, b) \sim (c, d) \iff a +_{\mathbb{N}} d = b +_{\mathbb{N}} c$$

Theorem

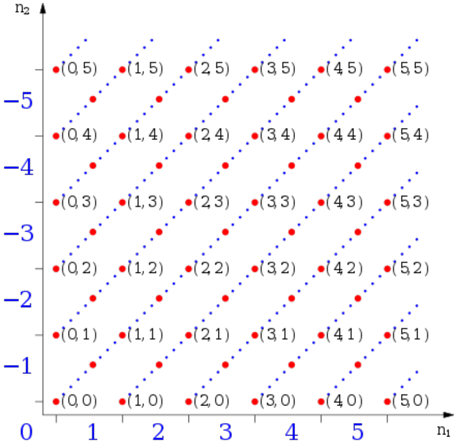
$\sim$ is an equivalence relation.

$$Q : \text{What is } \mathbb{N} \times \mathbb{N} / \sim ?$$

Definition ($\mathbb{Z}$)

$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim$$

$$[(1, 3)]_{\sim} = \{(0, 2), (1, 3), (2, 4), (3, 5), \cdots\} \triangleq -2 \in \mathbb{Z}$$



$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim$$

Definition ($+_{\mathbb{Z}}$)

$$[(m_1, n_1)] +_{\mathbb{Z}} [(m_2, n_2)] = [m_1 +_{\mathbb{N}} m2, n_1 +_{\mathbb{N}} n_2]$$

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Definition ($\cdot_{\mathbb{Z}}$)

$$[(m_1, n_1)] \cdot_{\mathbb{Z}} [(m_2, n_2)]$$
$$= [m_1 \cdot_{\mathbb{N}} m_2 +_{\mathbb{N}} n_1 \cdot_{\mathbb{N}} n_2, m_1 \cdot_{\mathbb{N}} n_2 +_{\mathbb{N}} n_1 \cdot_{\mathbb{N}} m_2]$$

Definition

$$\sim \subseteq (\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\}))^2$$

$$(a,b) \sim (c,d) \iff a \cdot_{\mathbb{Z}} d = b \cdot_{\mathbb{Z}} c$$

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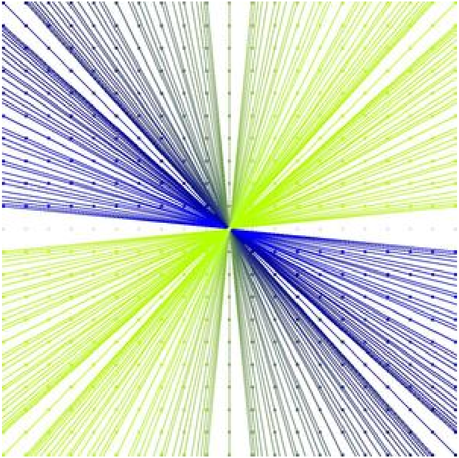
Definition ($\mathbb{Q}$)

$$\mathbb{Q} \triangleq \mathbb{Z} \times \mathbb{Z} / \sim$$



$$\mathbb{Q} \triangleq \mathbb{Z} \times \mathbb{Z} / \sim$$

How to define $\mathbb{R}$ as equivalence classes of ordered pairs of $\mathbb{Q}$?

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Thank
You!