2-3 Counting

魏恒峰

hfwei@nju.edu.cn

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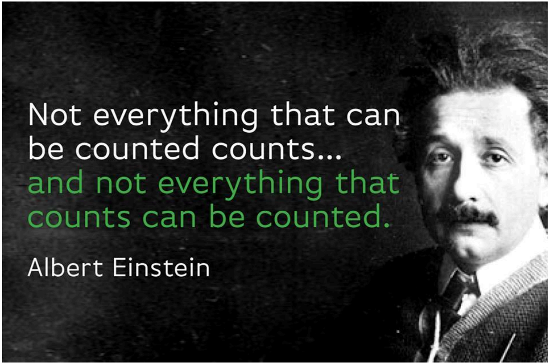
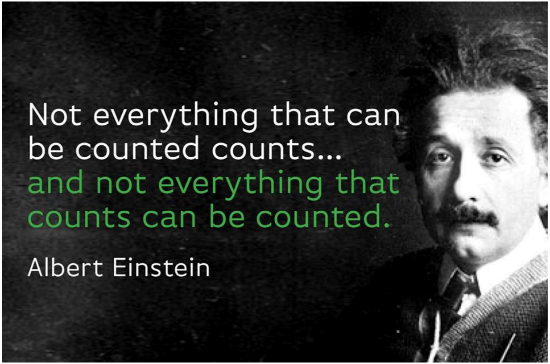

Not everything that can be counted counts...
and not everything that counts can be counted.
Albert Einstein



所以，学好“2-3 组合与计数”是多么重要!

Paring up (CS : $1.2 - 15$)

A tennis club has $2n$ members. We want to pair up the members by twos for singles matches.

(a) In how many ways can we pair up all the members of the club?

(b) Suppose that we also determine who serves first for each pairing. In how many ways can we specify our pairs?

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$$\frac{(2n)!}{2^n \cdot n!} \cdot 2^n = \frac{(2n)!}{n!}$$

Passing out Apples to Children

k-Permutation $(\text{CS} : 1.2 - 5)$

We need to pass out k distinct apples (pieces of fruit) to n children such that *each child may get at most one apple*.

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(b) What if $k > n$?

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$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

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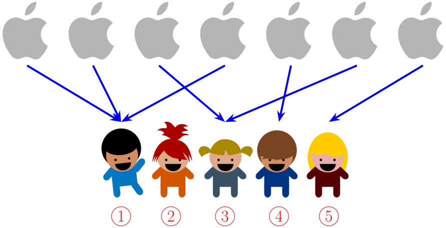

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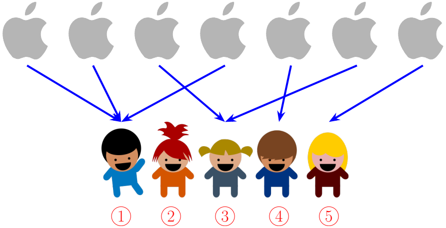

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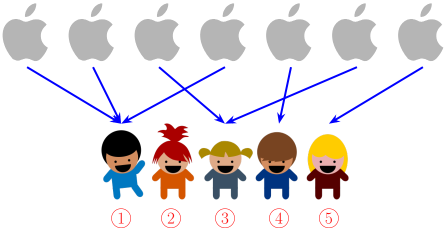

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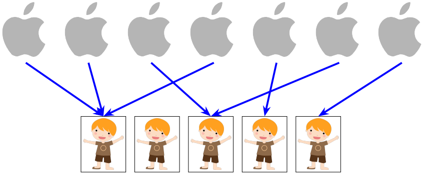

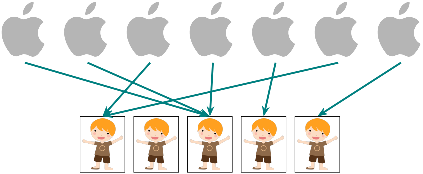

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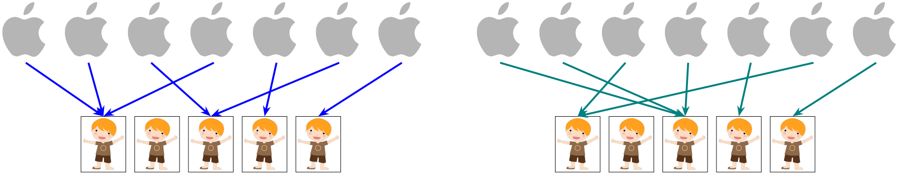

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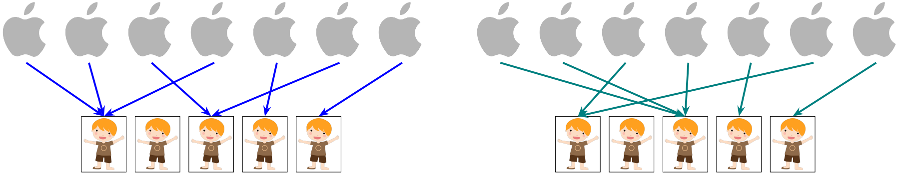

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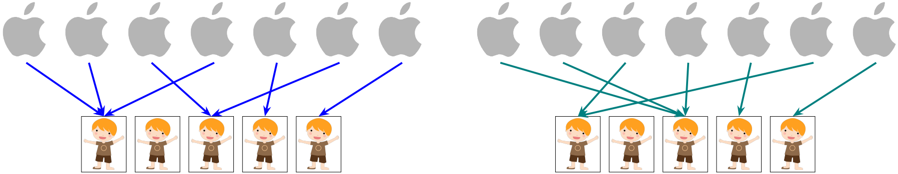

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Theorem ( )

$$p(k) \triangleq \sum_{x=1}^{x=k} p_x(k) \sim \frac{1}{4\sqrt{3}k} \exp\left(\pi\sqrt{\frac{2k}{3}}\right)$$

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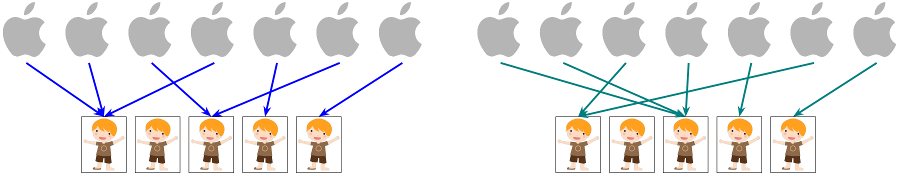

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Theorem (G. H. Hardy, Ramanujan (1918))

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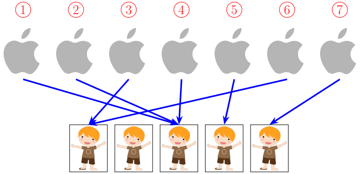

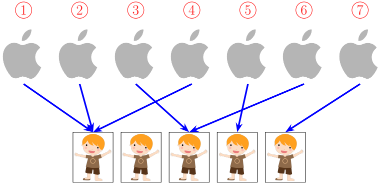

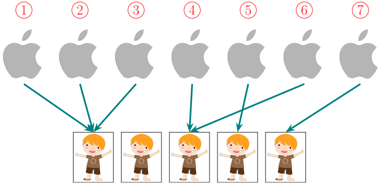

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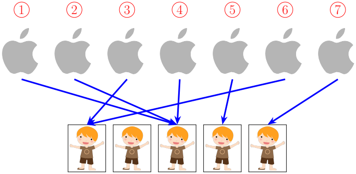

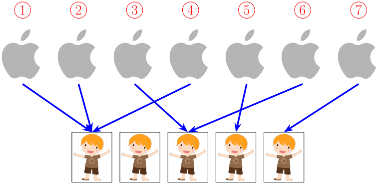

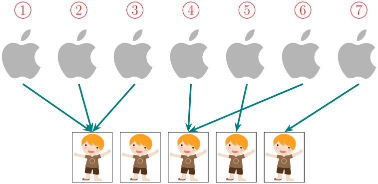

Set partition of $[1 \cdots k]$ into $\leq n$ parts

Set Partition (CS : $1.5 - 12$)

$$S(n,k) \ (\left\{ {n \atop k} \right\}) : \# \text{ of set partitions of } [1 \cdots n] \text{ into } k \text{ classes}$$

Set Partition $(\text{CS}: 1.5-12)$

$$S(n,k) \left(\begin{Bmatrix} n \\ k \end{Bmatrix}\right) : \#\text{ of set partitions of } [1 \cdots n] \text{ into } k \text{ classes}$$

Stirling number of the second kind

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Stirling number of the second kind

Theorem (Recurrence for $S(n,k)$)

$$S(0,0) = 1, \quad S(n,0) = S(0,n) = 0\ (n > 0)$$

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \quad n > 0, k > 0$$

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Proof.

$$S(n,k) = \underbrace{S(n-1,k-1)}_{n \text{ is alone}} + \underbrace{kS(n-1,k)}_{n \text{ is not alone}}$$

□

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Theorem (Berend & Tassa (2010))

$$B_n < \left(\frac{0.792n}{\ln(n+1)} \right)^n, n \in \mathbb{Z}^+$$

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Theorem (de Bruijn (1981))

As $n \to \infty$,

$$\frac{\ln B_n}{n} = \ln n - \ln\ln n - 1 + \frac{\ln\ln n}{\ln n} + \frac{1}{\ln n} + \frac{1}{2}\left(\frac{\ln\ln n}{\ln n}\right)^2 + O\left(\frac{\ln\ln n}{(\ln n)^2}\right)$$

THE TWELVEFOLD WAY

balls per urn	unrestricted	≤ 1	≥ 1
n labeled balls, m labeled urns	n-tuples of m things	n-permutations of m things	partitions of $\{1, \ldots, n\}$ into m ordered parts
n unlabeled balls, m labeled urns	n-multicombinations of m things	n-combinations of m things	compositions of n into m parts
n labeled balls, m unlabeled urns	partitions of $\{1, \ldots, n\}$ into $\leq m$ parts	n pigeons into m holes	partitions of $\{1, \ldots, n\}$ into m parts
n unlabeled balls, m unlabeled urns	partitions of n into $\leq m$ parts	n pigeons into m holes	partitions of n into m parts

Computing $\binom{n}{k}$ (CS $1.5:14$)

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procedure BINOM(n, k)                    ▷ Required: n ≥ k ≥ 0
    if k = 0 ∨ n = k then
        return 1
    return BINOM(n − 1, k) + BINOM(n − 1, k − 1)
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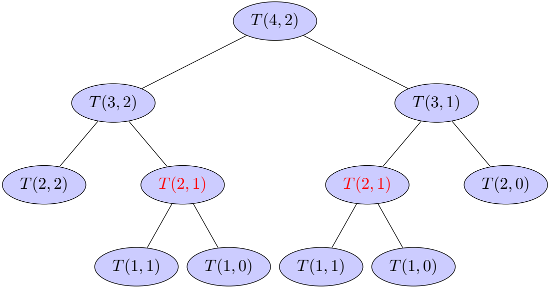

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$$\boxed{T(n,k) = \begin{cases} \\ T(n-1,k) + T(n-1,k-1) + c, \end{cases}}$$

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$$\alpha \binom{n}{0} - c = 0, \quad \alpha \binom{n}{n} - c = 0 \implies \alpha = c$$

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$$\alpha \binom{n}{0} - c = 0, \quad \alpha \binom{n}{n} - c = 0 \implies \alpha = c$$

$$\boxed{T(n,k) = c\binom{n}{k} - c}$$

7. 斐波那契数列的定义如下：$F_1=1, F_2=1, F_n=F_{n-1}+F_{n-2}\ (n\geq 3)$。如果用下面的函数计算斐波那契数列的第 n 项，则其时间复杂度为（　）。

```
int F(int n)
{
    if (n <= 2)
        return 1;
    else
        return F(n - 1) + F(n - 2);
}
```

A. $O(1)$　　B. $O(n)$　　C. $O(n^2)$　　D. $O(F_n)$

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时间复杂度 of what?

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

Q : How to calculate $\binom{5}{3}$?

procedure $\textsc{Binom}(n, k)$ ▷ Required: $n \geq k \geq 0$

    for $i \leftarrow 0$ **to** $n - k$ **do**

        $B[i][0] \leftarrow 1$

    for $i \leftarrow 1$ **to** k **do**

        $B[i][i] \leftarrow 1$

    for $j \leftarrow 1$ **to** k **do**

        for $d \leftarrow 1$ **to** $n - k$ **do**

            $i \leftarrow j + d$

            $B[i][j] \leftarrow B[i-1][j] + B[i-1][j-1]$

    return $B[n][k]$

procedure $\textsc{Binom}(n, k)$ ▷ Required: $n \geq k \geq 0$

  for $i \leftarrow 0$ **to** $n - k$ **do**

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    $B[i][i] \leftarrow 1$

  for $j \leftarrow 1$ **to** k **do**

    for $d \leftarrow 1$ **to** $n - k$ **do**

      $i \leftarrow j + d$

      $B[i][j] \leftarrow B[i-1][j] + B[i-1][j-1]$

  return $B[n][k]$

$$(n - k + 1) + (k) + k(n - k) = nk - k^2 + n + 1$$


Thank
You!