ON CANCELLATION IN GROUPS

R. HIRSHON, Polytechnic Institute of Brooklyn

Let $A \times B$ represent the direct product of the groups A and B. We shall say that B may be cancelled in direct products if

$$A \times B \approx A_1 \times B_1, \qquad B \approx B_1$$

imply $A \approx A_1$ for any A.

It seems natural to inquire about those groups which may be cancelled in direct products. We will show in this paper that a finite group B may be cancelled in direct products. As far as we can determine, this result does not appear in any standard text in group theory or algebra, perhaps because it appears to have been discovered as recently as 1947 ([4], introduction), and apparently is still not well known. Good use of it might be made, for example, in proving that the decomposition of a finite group as a direct product of indecomposable groups is unique up to isomorphism.

We present a proof of the cancellation theorem which we feel is the simplest available and is suitable for undergraduates. We also present in this paper an outline of a proof that an infinite cyclic group may not, in general, be cancelled in direct products, thus giving an example of the "simplest" type of group which may not be cancelled.

CANCELLATION THEOREM. *If B is a finite group, B may be cancelled in direct products.*

Proof. We observe first that it suffices to show

$$G = D \times B = D_1 \times B_1, \qquad B \approx B_1, \quad \text{imply } D \approx D_1. \tag{1}$$

We prove (1) by induction on $|B|$, the order of B.

Clearly (1) is true if $|B| = 1$. Assume (1) is true for groups B, with $|B| < k$. We prove (1) is true if $|B| = k$. First observe that if $B \cap D_1 = 1$ then $G = B \times D_1$, so that $D \approx G/B \approx D_1$. Hence, without loss of generality, we may assume $B \cap D_1 \neq 1$. Also by symmetry we may assume

$$F = B \cap D_1 \neq 1, \qquad K = B_1 \cap D \neq 1.$$

Now from (1), we may see

$$G/(F \times K) = (B \times D)/(F \times K) = (B_1 \times D_1)/(K \times F). \tag{2}$$

By a standard isomorphism theorem, we see from (2)

$$(B/F) \times (D/K) \approx (B_1/K) \times (D_1/F).$$

Hence, since $B \approx B_1$, we may write

$$B \times (B/F) \times (D/K) \approx B_1 \times (B_1/K) \times (D_1/F). \tag{3}$$

However,

$$B \times (B/F) \times (D/K) \approx [B \times (D/K)] \times B/F \approx [(B \times D)/K] \times B/F$$
$$= [(B_1 \times D_1)/K] \times B/F \approx (B_1/K) \times D_1 \times B/F.$$

In summary, we have

$$B \times (B/F) \times D/K \approx (B_1/K) \times D_1 \times B/F. \tag{4}$$

Note that our hypothesis is symmetrical in B and B_1 and D and D_1, so if we interchange B and B_1 and D and D_1 (and hence F and K), we see from (4)

$$B_1 \times (B_1/K) \times D_1/F \approx (B/F) \times D \times B_1/K. \tag{5}$$

Now note from (3) that the groups on the left hand sides of (4) and (5) are isomorphic. Consequently, the groups on the right hand sides of (4) and (5) are isomorphic; that is,

$$L_1 = D_1 \times (B/F) \times (B_1/K) \approx D \times (B/F) \times (B_1/K) = L_2. \tag{6}$$

Hence we may apply our inductive assumption twice in (6); that is, first cancel B_1/K in (6) and then cancel B/F. (To be quite precise, by using an isomorphism of L_1 onto L_2 obtained from (6), write (6) over as an equality between decompositions of L_2, and then apply the inductive assumption once, and then repeat this procedure again.) The result is $D_1 \approx D$, and the theorem is complete.

Kaplansky (in [5] p. 13) posed the following problem:

If B and B_1 are infinite cyclic abelian groups, and A is abelian and $A \times B \approx A_1 \times B_1$, is $A \approx A_1$? The question is answered affirmatively in [1], p. 55. It is surprising to discover that an infinite cyclic group may not be cancelled in general.

One can see the essential reason for this by considering a group H with the following properties:

(a) $H = \langle a \rangle L$, $L \cap \langle a \rangle = 1$, $L \triangle H$,

where $\langle a \rangle$ is an infinite cyclic group generated by a.

(b) There exists d, $d > 1$, such that a^d is in the centralizer of L.

(c) $K = \langle a^u \rangle L$ is not isomorphic to H, where u is an integer for which there exist integers s and e such that

(d) $eu - sd = \pm 1$.

Then if $\langle z \rangle$ is an infinite cyclic group and $G = \langle z \rangle \times H$ and if we set $w = z^e a^d$, $M = \langle z^s a^u \rangle L$, one can show $M \approx K$ and

$$G = \langle z \rangle \times H = \langle w \rangle \times M.$$

Since M and H are not isomorphic, this shows that an infinite cyclic group may not be cancelled in general. An example of such a group H is a group with two generators a and y, with defining relations $a^{-1}ya = y^4$, $y^{1024} = y$. One may take $L = \langle y \rangle$, $d = 5$, $u = 2$, $s = 1$, $e = 2$. We omit the proof that this group has the desired properties.

In closing, we point out that a group with a principal series, that is, one which obeys the ascending and descending chain condition for normal subgroups, may be cancelled in direct products. The proof is essentially the same as the one we have given for finite groups except that one uses induction on the length of a principal series. Some applications of this cancellation result appear in [2]. A sufficient condition for the cancellation of infinite groups which obey the maximal condition for normal subgroups, is given in [3].

References

1. L. Fuchs, Abelian Groups, Pergamon Press, New York, 1967.

2. R. Hirshon, Some Theorems on Hopficity. To appear. Transactions of the American Mathematical Society.

3. ——, Cancellation of groups with maximal condition. To appear in Proc. Amer. Math Soc.

4. B. Jónsson and A. Tarski, Direct Decompositions of Finite Algebraic Systems, Notre Dame Mathematical Lectures, Notre Dame, Indiana, 1947.

5. I. Kaplansky, Infinite Abelian Groups, The University of Michigan Press, Ann Arbor, 1962.

A MAXIMUM MODULUS PRINCIPLE FOR CLOSED ALGEBRAS OF LIPSCHITZ FUNCTIONS

R. M. CROWNOVER, University of Missouri

Let us call a complex-valued function f on a metric space (X, d) an *LOC function* if f satisfies a uniform Lipschitz condition on each compact subset of X, i.e., if for each compact $E\subset X$, there is a constant $K_E(f)$ such that for $x, y\in E$,

$$|f(x) - f(y)| \leqq K_E(f)d(x, y).$$

For example, each analytic function on a plane domain is an LOC function.

In general, uniform limits of LOC functions are not LOC functions. However, if the functions are analytic functions on a plane domain, then, of course, the uniform limits are again LOC functions. In the direction of a converse of this result, we shall obtain a maximum modulus theorem for certain algebras of LOC functions which are closed under uniform limits, and indeed obtain analyticity in one special case.

LEMMA. *Let A be a linear space of bounded functions on (X, d) which is closed in sup norm. If $E\subset X$, and each $f\in A$ satisfies a uniform Lipschitz condition on E, then there exists a constant K_E such that for any $f\in A$ with $\|f\|_\infty\leqq 1$,*

$$|f(x) - f(y)| \leqq K_E\, d(x, y) \qquad \text{for } x, y \in E. \tag{1}$$

Proof. Let $S=\{f:f\in A$ and such that for any $x, y\in E$, $|f(x)-f(y)|\leqq d(x,y)\}$. Then $A=\bigcup_{n=1}^\infty (nS)$; since A is a complete metric space, the Baire category theorem applies, implying for some n, the set $\overline{nS}=nS$ has nonvoid interior. Consequently for some $f_0\in S$, and $r>0$, $S\supset f_0+N(0; r)$, where $N(0; r)=\{h: h\in A$ and $\|h\|_\infty<r\}$. Since S is symmetric, $-f_0+N(0; r)\subset S$, and since S is convex, for each $h\in N(0; r)$, $h=\frac{1}{2}(-f_0+h)+\frac{1}{2}(f_0+h)$ lies in S, i.e., $N(0; r)\subset S$. It follows that if $K_E=1/r$, then (1) holds for all $x, y\in E$.

We now prove the aforementioned maximum modulus theorem.