Number-Theoretic Algorithms

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Number-Theoretic Algorithms

1. Modular Arithmetic
2. Euclid's Algorithm
3. Pairwise Relatively Prime
4. Chinese Remainder Theorem

Cancellation in modular arithmetic

(TC 31.4–2)

$$ad \equiv bd \pmod{n} \;\not\Longrightarrow\; a \equiv b \pmod{n}$$

$$ad \equiv bd \pmod{n}, d \perp n \;\Longrightarrow\; a \equiv b \pmod{n}$$

$$3 \cdot 2 \equiv 5 \cdot 2 \pmod{4} \quad 3 \not\equiv 5 \pmod{4}$$

Changing the modulus

$$3 \cdot 2 \equiv 5 \cdot 2 \pmod 4 \quad 3 \not\equiv 5 \pmod 4 \quad 3 \equiv 5 \pmod 2$$

$$ad \equiv bd \pmod{nd} \iff a \equiv b \pmod n \quad (d \neq 0)$$

$$\boxed{(a \bmod n)d = ad \bmod nd \quad \text{(distributive law)}}$$

$$ad \equiv bd \pmod n \iff a \equiv b \pmod{\frac{n}{(d,n)}}$$

Changing the modulus

$$n = n_1 n_2 \cdots n_k$$

$$a \equiv b \pmod{n} \implies a \equiv b \pmod{n_i}$$

$$a \equiv b \pmod{100} \implies a \equiv b \pmod{20} \implies a \equiv b \pmod{5}$$

Changing the modulus

$$n = n_1 n_2 \cdots n_k$$

$$a \equiv b \pmod{n_1}, a \equiv b \pmod{n_2} \iff a \equiv b \pmod{\operatorname{lcm}(n_1, n_2)}$$

$$a \equiv b \pmod{n_1}, a \equiv b \pmod{n_2} \iff a \equiv b \pmod{n_1 n_2}, \text{ if } n_1 \perp n_2$$

$$\forall 1 \le i \le k, a \equiv b \pmod{n_i} \iff a \equiv b \pmod{n}, \text{ if } n_i \perp n_j$$

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Worst-case analysis of Euclid's algorithm

(TC 31.2–5)

1. If $a > b \geq 0$, $\textsc{Euclid}(a, b)$ makes $\leq 1 + \log_{\phi} b$ recursive calls.

Lamé's theorem: $a > b \geq 1, b < F_{k+1} \implies r < k.$

$$k = 2 + \log_{\phi} b$$

To prove $b < F_{3+\log_{\phi} b}$.

$$F_k = \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}} > \frac{\phi^k - 1}{\sqrt{5}}$$

Worst-case analysis of Euclid's algorithm

(TC 31.2–5)

2. Improve this bound to $1 + \log_{\phi}(\frac{b}{(a,b)})$.

$$(a,b) = (a,b) \cdot (\frac{a}{(a,b)}, \frac{b}{(a,b)})$$

$$\begin{aligned} &(16,12) \\ &= (12,4) \\ &= (4,0) \\ &= 4 \end{aligned} \qquad \begin{aligned} &(4,3) \\ &= (3,1) \\ &= (1,0) \\ &= 1 \end{aligned}$$

$$\textsc{Euclid}(a,b) \leftrightarrow \textsc{Euclid}(\frac{a}{(a,b)}, \frac{b}{(a,b)})$$

Worst-case analysis of Euclid's algorithm

(TC 31.2–5)

2. Improve this bound to $1 + \log_\phi(\frac{b}{(a,b)})$.

$$\textsc{Euclid}(a,b) \leftrightarrow \textsc{Euclid}(\frac{a}{(a,b)}, \frac{b}{(a,b)})$$

$$\textsc{Euclid}(b, a \bmod b) \overset{?}{\leftrightarrow} \textsc{Euclid}(\frac{b}{(a,b)}, \frac{a}{(a,b)} \bmod \frac{b}{(a,b)})$$

$$\textsc{Euclid}(b, a \bmod b) \leftrightarrow \textsc{Euclid}(\frac{b}{(a,b)}, \frac{a \bmod b}{(a,b)})$$

$$\frac{a}{(a,b)} \bmod \frac{b}{(a,b)} = \frac{a \bmod b}{(a,b)}$$

Worst-case analysis of Euclid's algorithm

(TC 31.2–5)

2. Improve this bound to $1 + \log_{\phi}(\frac{b}{(a,b)})$.

Lemma (Generalization of Lemma 31.10)

If $a > b \geq 1, d = (a, b)$ and $\textsc{Euclid}(a, b)$ performs $k \geq 1$ recursive calls, then $a \geq dF_{k+2}$ and $b \geq dF_{k+1}$.

Average-case analysis of Euclid's algorithm

$$T(m,0) = 0; \quad T(m,n) = 1 + T(n, m \bmod n)\ n \geq 1$$

When m is chosen at random:

$$T_n = \frac{1}{n} \sum_{0 \leq k < n} T(k,n)$$

Assume that, for $0 \leq k < n$, $(n \bmod k)$ is "random":

$$T_n \approx 1 + \frac{1}{n}(T_0 + T_1 + \cdots + T_{n-1}) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = H_n \approx \ln n + O(1)$$

Reference

"The Art of Computer Programming, Vol 2: Seminumerical Algorithms (Section 4.5.3)" by Donald E. Knuth, 3rd edition.

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Pairwise relatively prime

(TC 31.2–9)

n_1, n_2, n_3, n_4 are pairwise relatively prime

$$\iff$$

$$\gcd(n_1 n_2, n_3 n_4) = \gcd(n_1 n_3, n_2 n_4) = 1$$

Pairwise relatively prime

(TC 31.2–9)

$n_1, n_2, \ldots, n_k$ are pairwise relatively prime

$$\iff$$

a set of $\lceil \lg k \rceil$ pairs of numbers derived from the n_i are relatively prime.

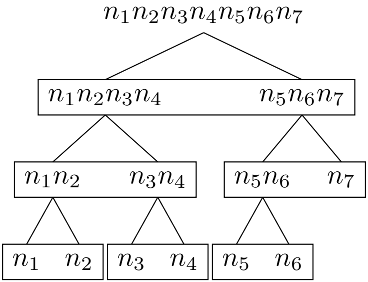
$$\binom{k}{2} = \Theta(k^2) \quad \text{(complete graph)}$$

$$\gcd(\boxed{1_L}, \boxed{1_R}) = \gcd(\boxed{2_L}, \boxed{2_R}) = \cdots = \gcd(\boxed{\lceil \lg k \rceil_L}, \boxed{\lceil \lg k \rceil_R}) = 1$$

$$k = 2: \quad \gcd(n_1, n_2) = 1$$

$$k = 3: \quad \gcd(n_1, n_2 n_3) = \gcd(n_2, n_3) = 1$$

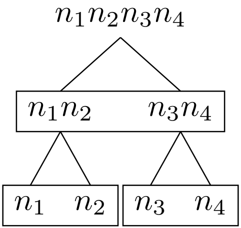
Pairwise relatively prime: divide-and-conquer



$$\begin{cases} T(1) = 0 \\ T(k) = T(\lceil \frac{k}{2} \rceil) + T(\lfloor \frac{k}{2} \rfloor) + 1 \end{cases} \implies T(k) = k - 1 = \Theta(k)$$

$$T_k = k - 1 : (n_i, n_{i+1}n_{i+2} \cdots n_k) \quad \forall 1 \le i < k$$

Pairwise relatively prime: smarter combination



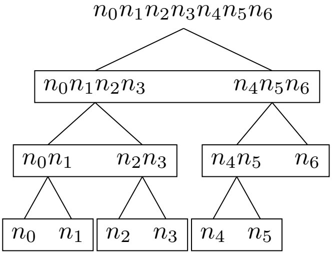
$$(n_1n_2, n_3n_4) = 1$$
$$(n_1, n_2) = 1,\ (n_3, n_4) = 1$$

$$(n_1n_2, n_3n_4) = 1$$
$$(n_1n_3, n_2n_4) = 1$$

$$\begin{cases} T(1) = 0 \\ T(k) = T(\lceil \frac{k}{2} \rceil) + 1 \end{cases} \implies T(k) = \lceil \lg k \rceil$$

Pairwise relatively prime: the dividing pattern

$$k = 7: \quad n_0, n_1, n_2, \ldots, n_6$$



$$0 : 000$$
$$1 : 001$$
$$2 : 010$$
$$3 : 011$$
$$4 : 100$$
$$5 : 101$$
$$6 : 110$$

$$T(k) = \lceil \lg k \rceil$$

Can we do even better?

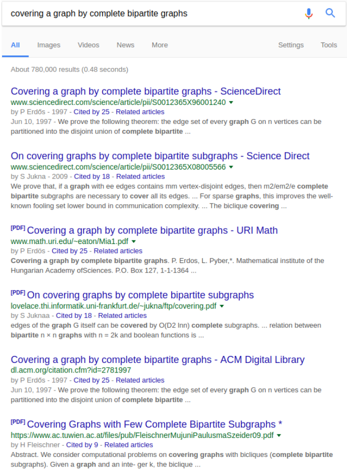
$$T(k) \geq \lceil \lg k \rceil$$

Prove by (strong) mathematical induction.

$$\begin{aligned} T(k) &\geq 1 + T(\lceil \frac{k}{2} \rceil) \\ &\geq 1 + \lceil \lg \lceil \frac{k}{2} \rceil \rceil \\ &= \lceil \lg k \rceil \end{aligned}$$

Biclique covering

Covering a complete graph with few complete bipartite subgraphs.



covering a graph by complete bipartite graphs

About 780,000 results (0.48 seconds)

Covering a graph by complete bipartite graphs - ScienceDirect
www.sciencedirect.com/science/article/pii/S0012365X96001240 ▾
by P Erdős - 1997 - Cited by 25 - Related articles
Jun 10, 1997 - We prove the following theorem: the edge set of every **graph** G on n vertices can be partitioned into the disjoint union of **complete bipartite** ...

On covering graphs by complete bipartite subgraphs - Science Direct
www.sciencedirect.com/science/article/pii/S0012365X08005566 ▾
by S Jukna - 2009 - Cited by 18 - Related articles
We prove that, if a **graph** with ee edges contains mm vertex-disjoint edges, then m2/em2/e **complete bipartite** subgraphs are necessary to **cover** all its edges. ... For sparse **graphs**, this improves the well-known fooling set lower bound in communication complexity. ... The biclique **covering** ...

[PDF] Covering a graph by complete bipartite graphs - URI Math
www.math.uri.edu/~eaton/Mia1.pdf ▾
by P Erdös - Cited by 25 - Related articles
Covering a graph by complete bipartite graphs. P. Erdos, L. Pyber,*. Mathematical institute of the Hungarian Academy ofSciences. P.O. Box 127, 1-1-1364 ...

[PDF] On covering graphs by complete bipartite subgraphs
lovelace.thi.informatik.uni-frankfurt.de/~jukna/ftp/covering.pdf ▾
by S Juknaa - Cited by 18 - Related articles
edges of the **graph** G itself can be **covered** by O(D2 lnn) **complete** subgraphs. ... relation between **bipartite** n × n **graphs** with n = 2k and boolean functions is ...

Covering a graph by complete bipartite graphs - ACM Digital Library
dl.acm.org/citation.cfm?id=2781997
by P Erdős - 1997 - Cited by 25 - Related articles
Jun 10, 1997 - We prove the following theorem: the edge set of every **graph** G on n vertices can be partitioned into the disjoint union of **complete bipartite** ...

[PDF] Covering Graphs with Few Complete Bipartite Subgraphs *
https://www.ac.tuwien.ac.at/files/pub/FleischnerMujuniPaulusmaSzeider09.pdf ▾
by H Fleischner - Cited by 9 - Related articles
Abstract. We consider computational problems on **covering graphs** with bicliques (**complete bipartite** subgraphs). Given a **graph** and an inte- ger k, the biclique ...

Biclique covering: rethinking the first divide-and-conquer

$$T(k) = k - 1$$

edge-disjoint biclique partition

Reference for $T(k) \geq k - 1$

"On the Addressing Problem for Loop Switching" by Graham and Pollak, 1971.

Reference for *weighted* biclique partition

"Covering a Graph by Complete Bipartite Graphs" by P. Erdős and L. Pyber, 1997.

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Chinese Remainder Theorem (CRT)

Theorem (CRT)

$$n_1, \ldots, n_k; \quad a_1, \ldots, a_k$$

$$n_i \perp n_j \quad i \neq j, \quad n = n_1 n_2 \cdots n_k$$

$$\exists! a \; (0 \leq a < n) : a \equiv a_i \pmod{n_i}.$$

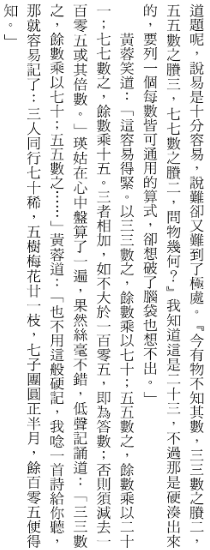
$$a \leftrightarrow (a_1, a_2, \ldots, a_k)$$

Proof for uniqueness.

$$a \equiv a' \pmod{n_i} \implies n \mid a - a'.$$

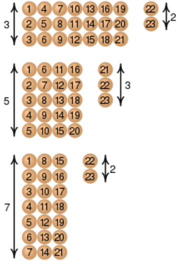
□

History of CRT



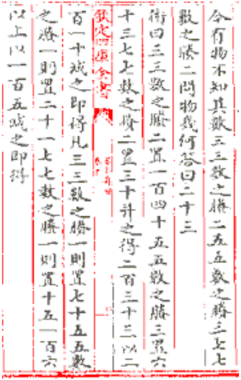
道題呢，說易是十分容易，說難卻又難到了極處。『今有物不知其數，三三數之賸二，五五數之賸三，七七數之賸二，問物幾何？』我知道這是二十三，不過那是硬湊出來的，要列一個每數皆可通用的算式，卻想破了腦袋也想不出。」

黃蓉笑道：「這容易得緊。以三三數之，餘數乘以七十；五五數之，餘數乘以二十一；七七數之，餘數乘十五。三者相加，如不大於一百零五，即為答數；否則須減去一百零五或其倍數。」瑛姑在心中盤算了一遍，果然絲毫不錯，低聲記誦道：「三三數之，餘數乘以七十；五五數之……」黃蓉道：「也不用這般硬記，我唸一首詩給你聽，那就容易記了：三人同行七十稀，五樹梅花廿一枝，七子團圓正半月，餘百零五便得知。」



“物不知数”

History of CRT



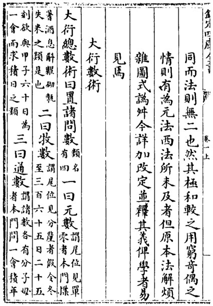
今有物不知其數三三數之賸二五五數之賸三七七數之賸二問物幾何答曰二十三

術曰三三數之賸二置一百四十五五數之賸三置六十三七七數之賸二置三十并之得二百三十三以二

欽定四庫全書

百一十減之即得凡三三數之賸一則置七十五五數之賸一則置二十一七七數之賸一則置十五一百六以上以一百五減之即得

“孙子算经”



欽定四庫全書　卷一上

同而法則無二也然其極和較之用窮奇偶之情則有為元法兩法所未及者但原本法解煩雜圖式譌舛今詳加改定並釋其義俾學者易見焉

大衍數術

大衍總數術曰置諸問數類名有四一曰元數謂尾位見單零者本門揲蓍酒息斛糶砌甎失米之類是也二曰收數謂尾位見分釐者假令冬至三百六十五日二十五刻欲與甲子六十日為一會而求積日之類三曰通數謂諸數各有分子母者本門問一會積年

秦九韶 “数书九章”
大衍求一术

Proof of CRT (1)

Nonconstructive proof.

$$f : [0, n) \to \prod_{1 \le i \le k} [0, a_i)$$

$$f : a \mapsto (a \bmod n_1, \ldots, a \bmod n_k)$$

- f is one-to-one.
- f is onto.

$$\exists a : f(a) = (a_1, \ldots, a_k).$$

□

Proof of CRT (2)

Constructive proof by induction.

$$a \equiv a_1 \pmod{n_1} \tag{1}$$

$$a \equiv a_2 \pmod{n_2} \tag{2}$$

$$(1) \implies a = a_1 + n_1 y$$

$$x = a_1 + n_1 n_1^{-1}(a_2 - a_1) \pmod{n_1 n_2}$$

□

Proof of CRT (3)

Constructive proof by induction.

$$a \equiv a_1 \pmod{n_1} \tag{3}$$

$$a \equiv a_2 \pmod{n_2} \tag{4}$$

$$n_1 \perp n_2 \implies n_1 n_1' + n_2 n_2' = 1$$

$$x = a_1 n_1 n_1' + a_2 n_2 n_2' \pmod{n_1 n_2}$$

□

Proof of CRT (4)

Constructive proof.

1. $x \equiv 1 \pmod{n_i}, \quad x \equiv 0 \pmod{n_j}\ (i \neq j)$

$$x = M_i(M_i^{-1} \bmod n_i) \implies x = M_i M_i^{-1} \pmod{n}$$

2. $x \equiv a_i \pmod{n_i}, \quad x \equiv 0 \pmod{n_j}\ (i \neq j)$

$$x = a_i M_i M_i^{-1} \pmod{n}$$

3. $a \equiv a_i \pmod{n_i}, \forall 1 \leq i \leq k$

$$a = \sum_{1 \leq i \leq k} a_i M_i M_i^{-1} \pmod{n}$$

□

Proof of CRT (5)

More efficient constructive proof.

Reference

“The Residue Number System” by Garner, 1959.

Reference

“The Art of Computer Programming, Vol 2: Seminumerical Algorithms (Section 4.3.2)” by Donald E. Knuth, 3rd edition.

□

Operations over CRT

$$a \leftrightarrow (a_1, a_2, \dots, a_n)$$

$$a \pm b \leftrightarrow (a_1 \pm b_1, a_2 \pm b_2, \dots, a_n \pm b_n)$$
$$a \times b \leftrightarrow (a_1 \times b_1, a_2 \times b_2, \dots, a_n \times b_n)$$

TC 31.5–3

$$a \leftrightarrow (a_1, a_2, \dots, a_n), (a, n) = 1 \implies a^{-1} \leftrightarrow (a_1^{-1}, a_2^{-1}, \dots, a_n^{-1})$$

Proof.

$$a^{-1} \equiv a_i^{-1} \pmod{n_i} \impliedby \begin{cases} a \equiv a_i \pmod{n_i} \\ (a, n) = 1 \end{cases}$$

□

The ϕ function

Theorem (The ϕ function)

$$\phi(p) = p - 1$$
$$\phi(p^k) = p^k - p^{k-1}$$

$$\phi(n) = n \prod_{i=1}^{r} (1 - \frac{1}{p_i}) \quad (n = \prod_{i=1}^{r} {p_i}^{k_i})$$

"We shall not prove this formula here." — *CLRS (Section 31.3)*

Let us prove this formula now.

$$m \perp n \implies \phi(mn) = \phi(m)\phi(n)$$

The ϕ function

Theorem (The ϕ function)

$$m \perp n \implies \phi(mn) = \phi(m)\phi(n)$$

Proof.

$$U_{mn} = \{a \bmod mn, (a, mn) = 1\}$$
$$U_m = \{b \bmod m, (b, m) = 1\} \quad U_n = \{c \bmod n, (c, n) = 1\}$$

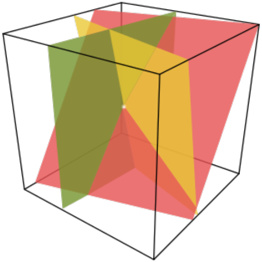
$$f : U_{mn} \to U_m \times U_n$$
$$f(a \bmod mn) = (a \bmod m, a \bmod n).$$

□

Secret sharing using the CRT

Definition ((k, n)-threshold secret sharing scheme)

$(3, 3)$-secret sharing:



Reference

“How to Share a Secret” by Maurice Mignotte, 1982.

Secret sharing using the CRT

1. Choose m_i:

$$m_1 < m_2 < \cdots < m_n, \quad m_i \perp m_j, \quad \prod_{i=n-k+2}^{n} m_i < \prod_{i=1}^{k} m_i$$

2. Choose the secret S:

$$\prod_{i=n-k+2}^{n} m_i < S < \prod_{i=1}^{k} m_i$$

3. Compute the shares:

$$s_i = S \bmod m_i$$

Solving simultaneous congruences

(TC 31.5–2)

$$\begin{cases} x \equiv 1 \pmod{9} \\ x \equiv 2 \pmod{8} \\ x \equiv 3 \pmod{7} \end{cases}$$

$$x \equiv 10 \pmod{504}$$

Solving simultannous congruences

CRT with large modulus

$$19x \equiv 556 \pmod{1155}$$

$$\begin{cases} 19x \equiv 556 \pmod{3} \\ 19x \equiv 556 \pmod{5} \\ 19x \equiv 556 \pmod{7} \\ 19x \equiv 556 \pmod{11} \end{cases}$$

$$\begin{cases} x \equiv 1 \pmod{3} \\ x \equiv 4 \pmod{5} \\ x \equiv 2 \pmod{7} \\ x \equiv 9 \pmod{11} \end{cases}$$

Solving simultaneous congruences

CRT with non-pairwisely co-prime moduli

$$\begin{cases} x \equiv 3 \pmod{8} \\ x \equiv 11 \pmod{20} \\ x \equiv 1 \pmod{15} \end{cases}$$

$$\begin{cases} x \equiv 3 \pmod{2^3} \end{cases} \qquad \begin{cases} x \equiv 3 \pmod{2^2} \\ x \equiv 1 \pmod{5} \end{cases} \qquad \begin{cases} x \equiv 1 \pmod{3} \\ x \equiv 1 \pmod{5} \end{cases}$$

$$\begin{cases} x \equiv 3 \pmod{2^3} \\ x \equiv 3 \pmod{2^2} \end{cases} \qquad \begin{cases} x \equiv 1 \pmod{3} \end{cases} \qquad \begin{cases} x \equiv 1 \pmod{5} \\ x \equiv 1 \pmod{5} \end{cases}$$

Solving simultaneous congruences

Theorem (CRT with non-pairwisely coprime moduli)

$$a_i \equiv a_j \pmod{(n_i, n_j)}$$

$$0 \leq a < \mathrm{lcm}(n_1, n_2, \ldots, n_k)$$

Simultaneous incongruences

$$\exists ? a, \forall 1 \le i \le k : a \not\equiv a_i \pmod{n_i}$$

NP-complete

THANK YOU