计算机问题求解 – 论题2-13

- 动态规划

课程研讨

- TC第15章

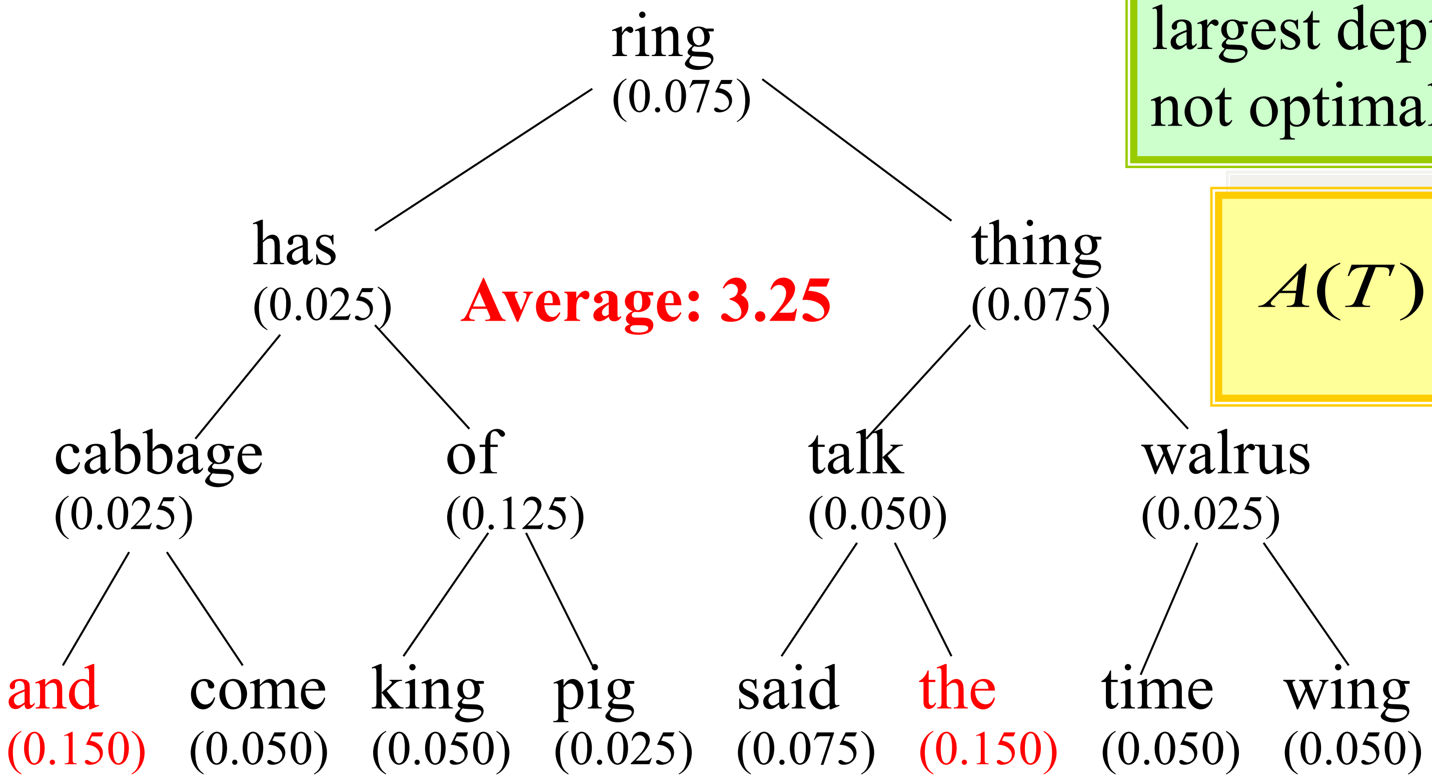
问题0：dynamic programming的基本概念

- 什么样的问题可以使用dynamic programming来求解？它高效的根本原因是什么？付出了什么代价？
- 你理解dynamic programming的四个步骤了吗？
  1. Characterize the structure of an optimal solution.
  2. Recursively define the value of an optimal solution.
  3. Compute the value of an optimal solution.
  4. Construct an optimal solution from computed information.

- 广义上决定dynamic programming运行时间的要素是哪两点？
- top-down with memorization和bottom-up method哪个更快？

问题1：Keys with Different Frequencies

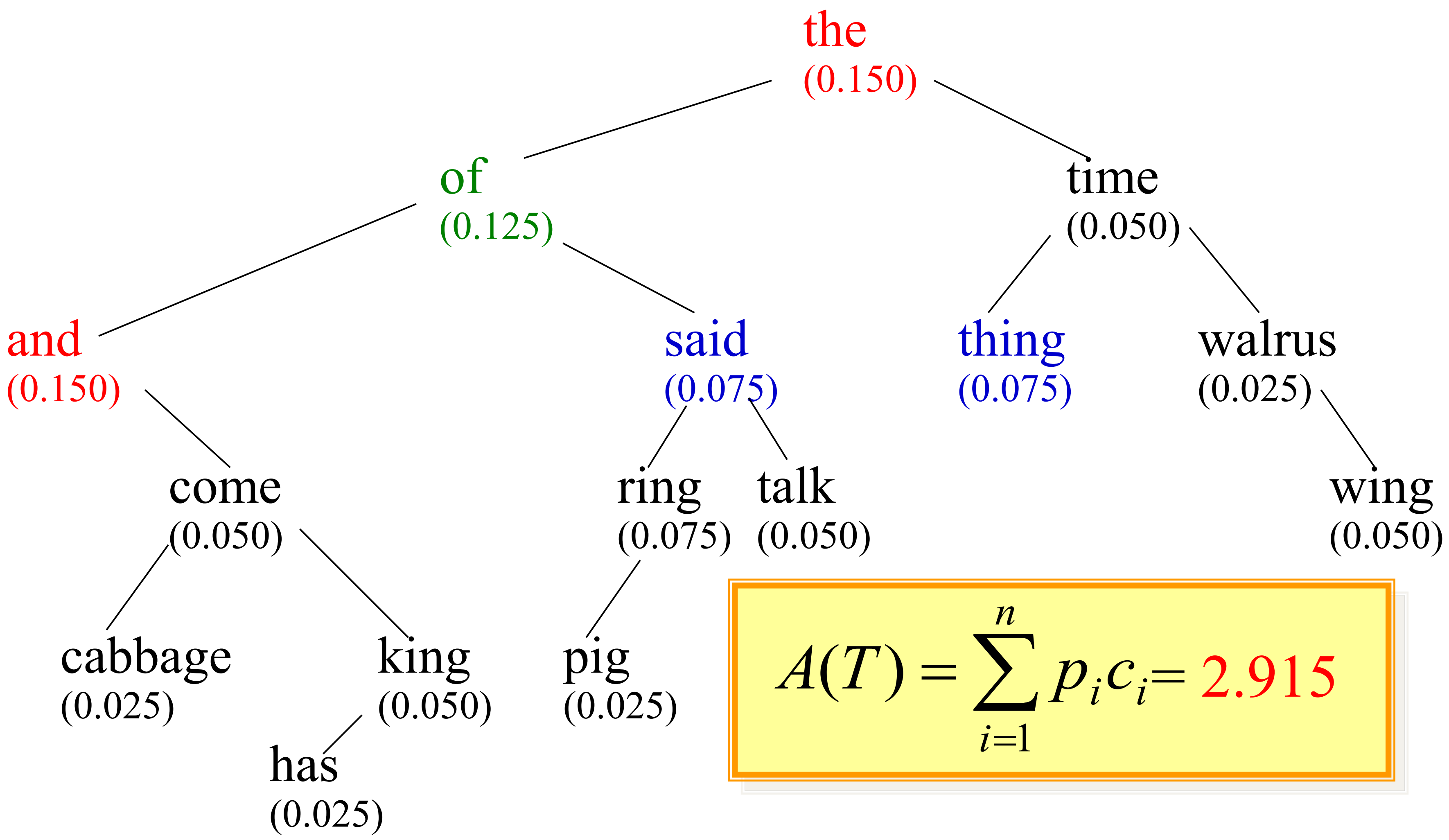
A binary search tree perfectly balanced



Since the keys with largest frequencies have largest depth, this tree is not optimal.

$$A(T) = \sum_{i=1}^{n} p_i c_i$$

Improved for a Better Average



$$\boxed{A(T) = \sum_{i=1}^{n} p_i c_i = 2.915}$$

矩阵连乘的问题

- 需要完成的任务:

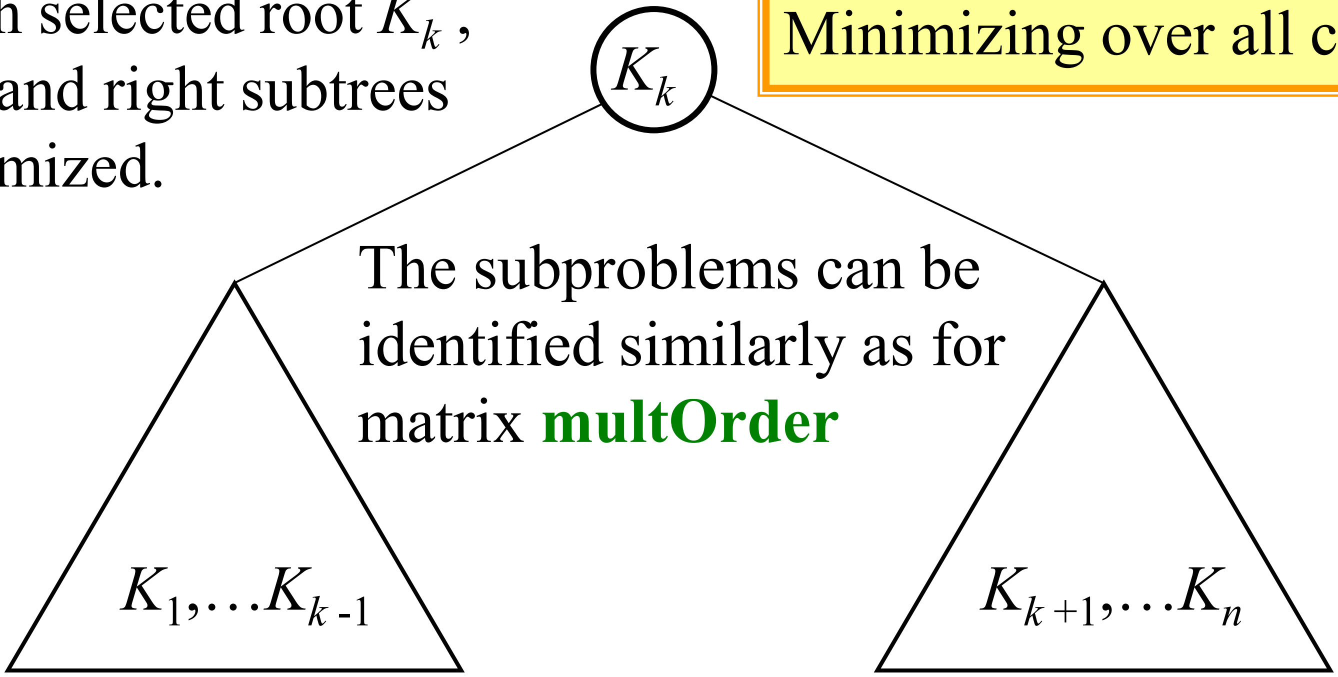
  求乘积: $A_1 \times A_2 \times \ldots \times A_{n-1} \times A_n$

  A_i 是二维矩阵，一般不是方阵，大小符合乘法规定的要求。
- 为什么会成为问题:
  - 矩阵乘法满足结合律，因此我们可以任意指定运算顺序;
  - 而不同的计算顺序代价差别很大。
- 优化问题: **什么样的次序计算代价最小?**

Plan of Optimal Binary Tree

For each selected root K_k , the left and right subtrees are optimized.

The problem is decomposes by the choices of the root. Minimizing over all choices



The subproblems can be identified similarly as for matrix **multOrder**

Subproblems as left and right subtrees

Problem Rephrased

- Subproblem identification
  - The keys are in sorted order.
  - Each subproblem can be identified as a pair of index (low, high)
- Expected solution of the subproblem
  - For each key K_i, a weight p_i is associated.

    Note: p_i is the probability that the key is searched for.
  - The subproblem (low, high) is to find the binary search tree with *minimum weighted retrieval cost*.

Minimum Weighted Retrieval Cost

- A(low, high, r) is the minimum weighted retrieval cost for subproblem (low, high) when K_r is chosen as the root of its binary search tree.
- A(low, high) is the minimum weighted retrieval cost for subproblem (low, high) over all choices of the root key.
- p(low, high), equal to $p_{\text{low}}+p_{\text{low+1}}+\ldots+p_{\text{high}}$, is the weight of the subproblem (low, high).

  Note: p(low, high) is the probability that the key searched for is in this interval .

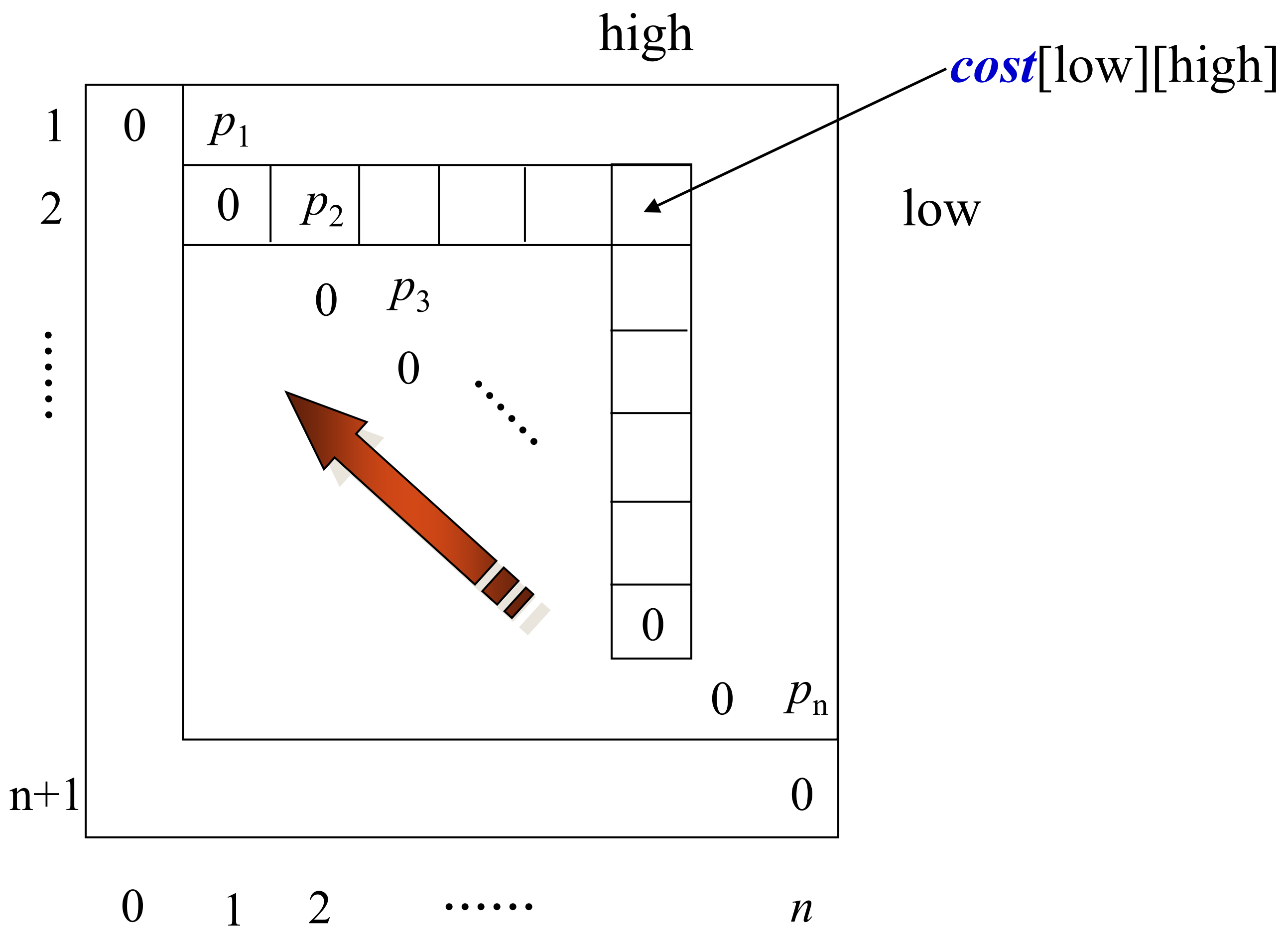
Integrating Solutions of Subproblem

- Weighted retrieval cost of a subtree
  - Let T is a particular tree containing $K_{\text{low}}, \ldots, K_{\text{high}}$, the weighted retrieval cost of T is W, with T being a whole tree. Then, as a subtree with the root at level 1, the weighted retrieval cost of T will be: $\boldsymbol{W+p(\text{low, high})}$
- So, the recursive relations:
  - $A(\text{low, high}, r)$
    $= p_r + p(\text{low}, r-1) + A(\text{low}, r-1) + p(r+1, \text{high}) + A(r+1, \text{high})$
    $= p(\text{low, high}) + A(\text{low}, r-1) + A(r+1, \text{high})$
  - $A(\text{low, high}) = \min\{A(\text{low, high}, r) \mid \text{low} \le r \le \text{high}\}$

Avoiding Repeated Work by Storing

- Array *cost*: *cost*[low][high] gives the minimum weighted search cost of subproblem (low,high).
- Array *root*: *root*[low][high] gives the best choice of root for subproblem (low,high)
- The *cost*[low][high] depends upon subproblems with higher first index(row number) and lower second index(column number)

Computation of the Array *cost*



Optimal BST: DP Algorithm

```
bestChoice(prob, cost, root, low, high)
    if (high<low)
        bestCost=0;
        bestRoot=-1;
    else
        bestCost=∞;
    for (r=low; r≤high; r++)
        rCost=p(low,high)+cost[low][r-1]+cost[r+1][high];
        if (rCost<bestCost)
            bestCost=rCost;
            bestRoot=r;
        cost[low][high]=bestCost;
        root[low][high]=bestRoot;
    return
```

```
optimalBST(prob,n,cost,root)
    for (low=n+1; low≥1; low--)
        for (high=low-1; high≤n; high++)
            bestChoice(prob,cost,root,low,high)
    return cost
```

in $\Theta(n^3)$

问题2：dynamic programming的实例

- 你能说明求解rod cutting的四个步骤吗？
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  2. Recursively define the value of an optimal solution.
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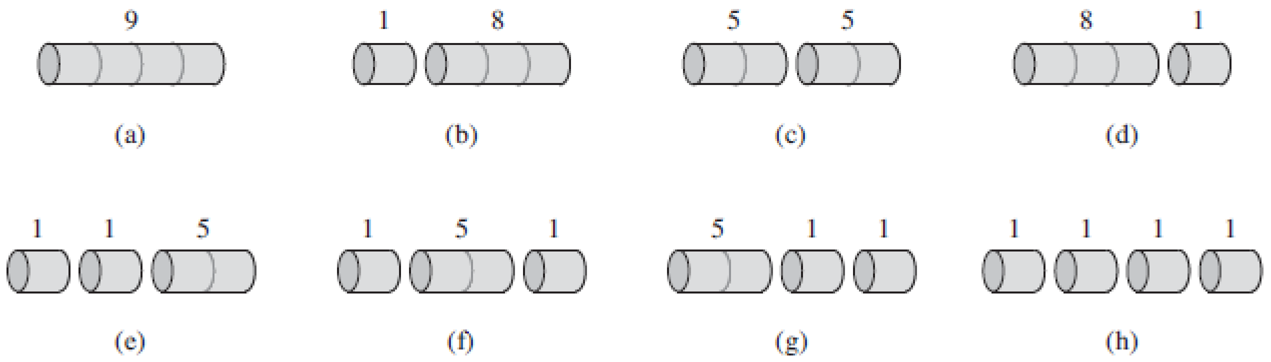

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

问题2：Separating Sequence of Words

- Word-length $w_1, w_2, \ldots, w_n$ and line-width: W
- Basic constraint: if $w_\mathrm{i}, w_{\mathrm{i}+1}, \ldots, w_\mathrm{j}$ are in one line, then $w_\mathrm{i}+w_{\mathrm{i}+1}+ \ldots+w_\mathrm{j}\leq W$
- Penalty for one line: some function of X. X is:
  - 0 for the last line in a paragraph, and
  - $W-(w_\mathrm{i}+w_{\mathrm{i}+1}+ \ldots+w_\mathrm{j})$ for other lines
- The problem
  - how to separate a sequence of words(forming a paragraph) into lines, making the penalty of the paragraph, which is the sum of the penalties of individual lines, minimized.

Solution by Greedy Strategy

i	word	*w*
1	Those	6
2	who	4
3	cannot	7
4	remember	9
5	the	4
6	past	5
7	are	4
8	condemned	10
9	to	3
10	repeat	7
11	it.	4

W is 17, and penalty is X^3

Solution by greedy strategy

words	(1,2,3)	(4,5)	(6,7)	(8,9)	(10,11)
X	0	4	8	4	0
penalty	0	64	512	64	0

Total penalty is **640**

An improved solution

words	(1,2)	(3,4)	(5,6,7)	(8,9)	(10,11)
X	7	1	4	4	0
penalty	343	1	64	64	0

Total penalty is **472**

Problem Decomposition

- Representation of subproblem: a pair of indexes (i,j), breaking words i through j into lines with minimum penalty.
- Two kinds of subproblem
  - (k, n): the penalty of the last line is 0
  - all other subproblems
- For some k, the combination of the optimal solution for $(1,k)$ and $(k+1,n)$ gives an optimal solution for $(1,n)$.
- Subproblem graph
  - About n^2 vertices
  - Each vertex (i,j) has an edge to about $j-i$ other vertices, so, the number of edges is in $\Theta(n^3)$

Simpler Identification of subproblem

- If a subproblem concludes the paragraph, then (k,n) can be simplified as (k). There are about k subproblems like this.
- Can we eliminate the use of (i,j) with $j<n$?
  - Put the first k words in the first line(with the basic constraint satisfied), the subproblem to be solved is $(k+1,n)$
  - Optimizing the solution over all k's. (k is at most $W/2$)

Breaking Sequence into lines

```
lineBreak($w,W,i,n,L$)
    if ($w_i + w_{i+1} + \ldots + w_n \leq W$)
        <Put all words on line $L$, set penalty to 0>
    else
        for (k=1; $w_i + \ldots + w_{i+k-1} \leq W$; k++)
            $X = W - (w_i + \ldots + w_{i+k-1})$;
            kPenalty=lineCost($X$)+lineBreak($w,W, i+k, n, L+1$)
            <Set penalty always to the minimum kPenalty>
            <Updating $k_{min}$, which records the k that produced
                                      the minimum penalty>
            <Put words $i$ through $i+k_{min}-1$ on line $L$>
    return penalty
```

In *DP* version this is replaced by "**Recursion or Retrieve**"

In DP version, "**Storing**" inserted

Analysis of lineBreak

- Since each subproblem is identified by only one integer k, for (k,n), the number of vertex in the subproblem is at most n.
- So, in $\mathcal{DP}$ version, the recursion is executed at most n times.
- The loop is executed at most $W/2$ times.
- So, the running time is in $\Theta(Wn)$. In fact, W, the line width, is usually a constant. So, $\boldsymbol{\Theta(n)}$.
- The extra space for the dictionary is in $\boldsymbol{\Theta(n)}$.

问题3：Longest Common Subsequence

- 你能说明求解longest common subsequence的四个步骤吗？
  1. Characterize the structure of an optimal solution.
  2. Recursively define the value of an optimal solution.
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$$S_1 = \mathrm{ACCGGTCGAGTGCGCGGAAGCCGGCCGAA}$$

$$S_2 = \mathrm{GTCGTTCGGAATGCCGTTGCTCTGTAAA}$$

$$\mathrm{GTCGTCGGAAGCCGGCCGAA}$$

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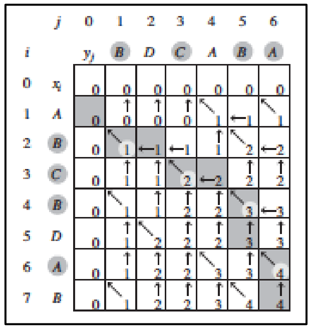
$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0\,, \\ c[i-1,j-1]+1 & \text{if } i,j > 0 \text{ and } x_i = y_j\,, \\ \max(c[i,j-1], c[i-1,j]) & \text{if } i,j > 0 \text{ and } x_i \neq y_j\,. \end{cases}$$



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问题4：Longest palindrome subsequence

- 你能说明求解longest palindrome subsequence的四个步骤吗？
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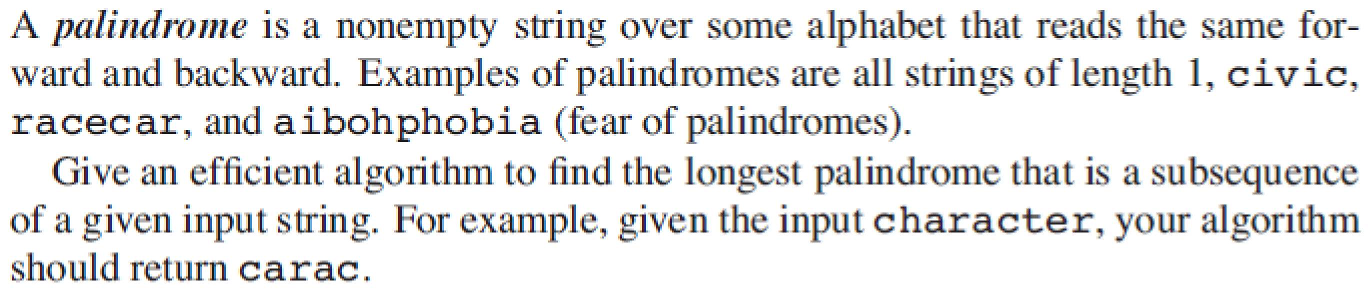

A ***palindrome*** is a nonempty string over some alphabet that reads the same forward and backward. Examples of palindromes are all strings of length 1, `civic`, `racecar`, and `aibohphobia` (fear of palindromes).

Give an efficient algorithm to find the longest palindrome that is a subsequence of a given input string. For example, given the input `character`, your algorithm should return `carac`.

问题4：Longest palindrome subsequence

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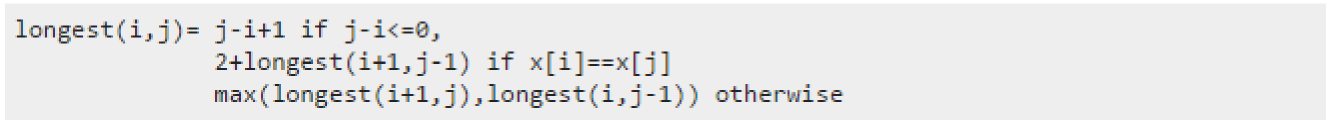
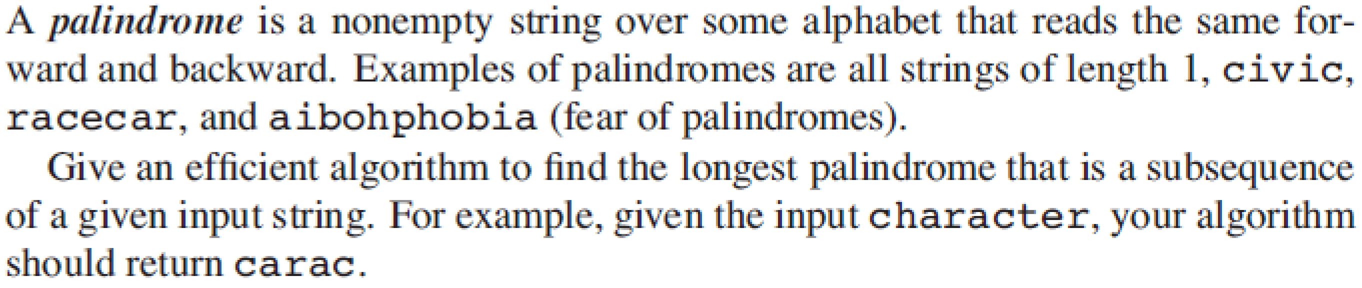


```
longest(i,j)= j-i+1 if j-i<=0,
              2+longest(i+1,j-1) if x[i]==x[j]
              max(longest(i+1,j),longest(i,j-1)) otherwise
```



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问题5：Edit distance

- 你能说明求解edit distance的四个步骤吗？
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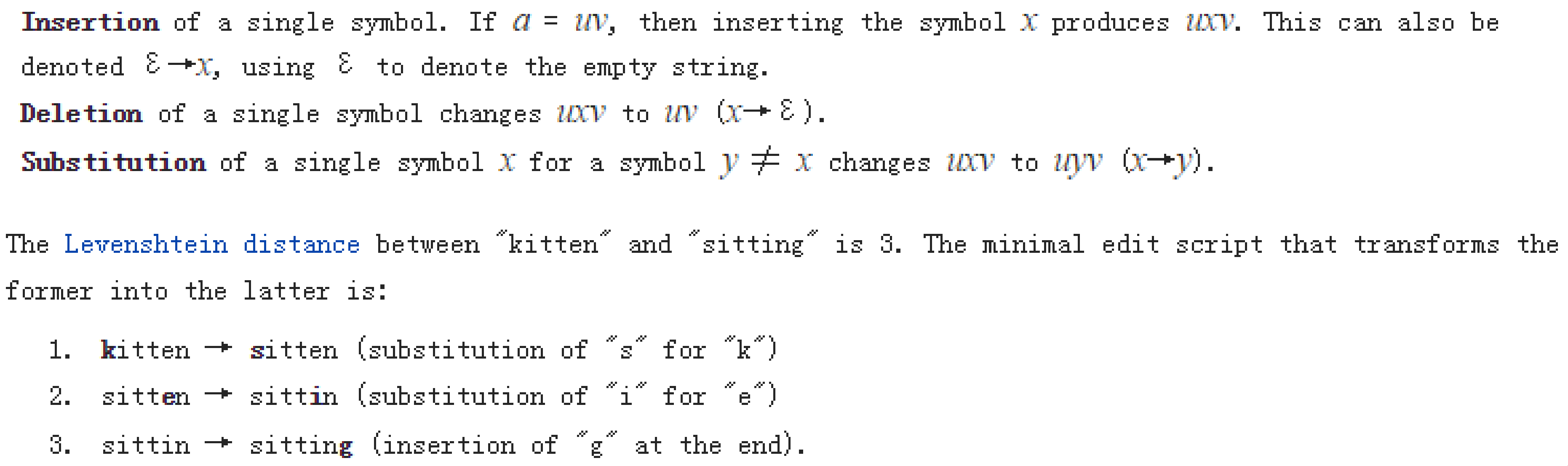

Insertion of a single symbol. If $a = uv$, then inserting the symbol x produces uxv. This can also be denoted $\varepsilon \rightarrow x$, using ε to denote the empty string.

Deletion of a single symbol changes uxv to uv ($x \rightarrow \varepsilon$).

Substitution of a single symbol x for a symbol $y \neq x$ changes uxv to uyv ($x \rightarrow y$).

The Levenshtein distance between "kitten" and "sitting" is 3. The minimal edit script that transforms the former into the latter is:

1. **k**itten → **s**itten (substitution of "s" for "k")
2. sitt**e**n → sitt**i**n (substitution of "i" for "e")
3. sittin → sittin**g** (insertion of "g" at the end).

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$$d_{ij} = \begin{cases} d_{i-1,j-1} & \text{for } a_j = b_i \\ \min \begin{cases} d_{i-1,j} + w_{\text{del}}(b_i) \\ d_{i,j-1} + w_{\text{ins}}(a_j) \\ d_{i-1,j-1} + w_{\text{sub}}(a_j, b_i) \end{cases} & \text{for } a_j \neq b_i \end{cases} \quad \text{for } 1 \leq i \leq m, 1 \leq j \leq n.$$

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问题6： Subset Sum

- Given a set $A=\{s_1,s_2,\ldots, s_n\}$, where s_i (for $i=1,2,\ldots, n$) is a natural number, and a natural number S, determine whether there is a subset of A totaling exactly S. Design a dynamic programming algorithm for solving the problem.

Decomposition the Problem

- Suppose subset $A_i \in A$ is a solution of the problem and $s_j \in A_i$, then we have

$$\sum(A_i - \{s_j\}) = S - s_j$$

- Thus, the problem can be divided into several stages, in each of which one element is found.
- States: all the possible values of subset sum in each stages.

Basic Idea

- Using a two-dimension boolean table T, in which $T[i, j]$=**true** if and only if there is a subset of the first i items of A totaling exactly j.

- Initialization
    - For each elements in $T[n+1][S]$, set as false
- Main loop to calculate each value

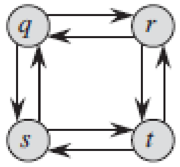
Main loop

```
for (i = 0; i <= n +1; i ++)
   if (A[i] == S ) return true;
   else if (A[i] < S) T[i, A[i]] = true;
   for (j = 0; j <= S + 1; j ++)
      if (T[i -1, j] )
      {
        T[i, j] = true;
        if ((T[i, j] + A[i]) == S) return true;
        else if ((t = (T[i, j] + A[i])) < S) T[i, t] = true;
      }
return false;
```

Time $O(nS)$ Space $O(nS)$

问题7： dynamic programming的实例 (续)

- unweighted longest simple path为什么不具有最优子结构？
- unweighted shortest simple path为什么不存在这个问题？



计算机问题求解 – 论题2-14

- 贪心算法

  课程研讨

  - TC第16.1-16.3节、第17章

问题1： greedy algorithms (续)

Describe an efficient algorithm that, given a set $\{x_1, x_2, \ldots, x_n\}$ of points on the real line, determines the smallest set of unit-length closed intervals that contains all of the given points. Argue that your algorithm is correct.

问题1： greedy algorithms (续)

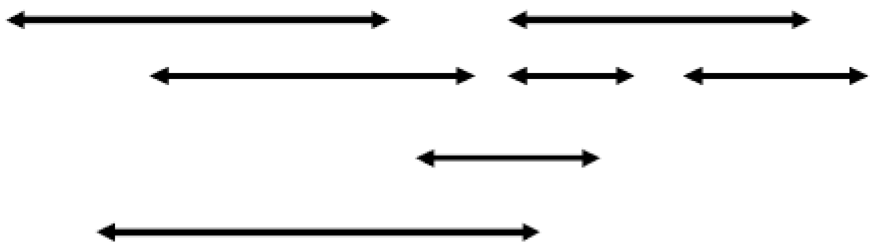
Describe an efficient algorithm that, given a set $\{x_1, x_2, \ldots, x_n\}$ of points on the real line, determines the smallest set of unit-length closed intervals that contains all of the given points. Argue that your algorithm is correct.

Sol: First we sort the set of n points $\{x_1, x_2, ..., x_n\}$ to get the set $Y = \{y_1, y_2, ..., y_n\}$ such that $y_1 \leq y_2 \leq ... \leq y_n$. Next, we do a linear scan on $\{y_1, y_2, ..., y_n\}$ started from y_1. Everytime while encountering y_i, for some $i \in \{1, ..., n\}$, we put the closed interval $[y_i, y_i + 1]$ in our optimal solution set S, and remove all the points in Y covered by $[y_i, y_i + 1]$. Repeat the above procedure, finally output S while Y becomes empty. We next show that S is an optimal solution.

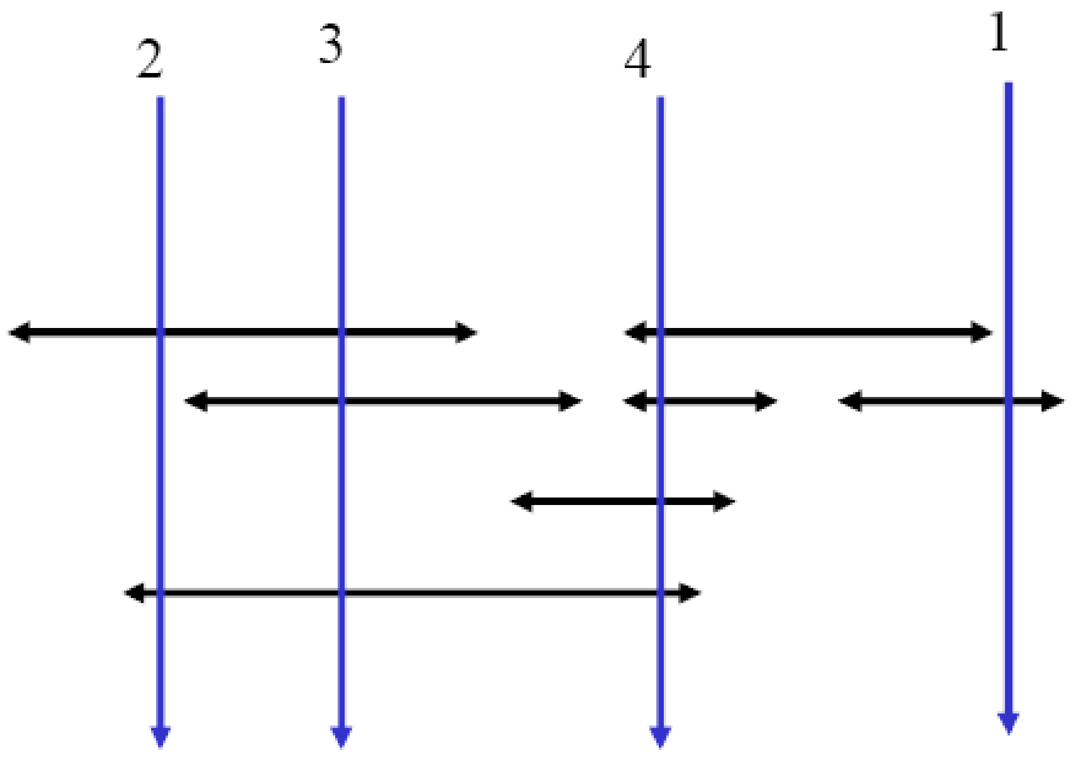
We claim that there is an optimal solution which contains the unit-length interval $[y_1, y_1 + 1]$. Suppose that there exists an optimal solution S^* such that y_1 is covered by $[x', x'+1] \in S^*$ where $x' < 1$. Since y_1 is the leftmost element of the given set, there is no other point lying in $[x', y_1)$. Therefore, if we replace $[x', x'+1]$ in S^* by $[y_1, y_1+1]$, we will get another optimal solution. This proves the claim and thus explains the greedy choice property. Therefore, by solving the remaining subproblem after removing all the points lying in $[y_1, y_1 + 1]$, that is, to find an optimal set of intervals, denoted as S', which cover the points to the right of $y1 + 1$, we will get an optimal solution to the original problem by taking union of $[y_1, y_1 + 1]$ and S'.

Scheduling Activities into Rooms

- **Instance**: a set of n classes, each with start time s_i and finishing time f_i
- **Problem**: find the minimum number of classrooms required to hold all classes



Depth of the Graph



A (bad) Room Scheduling Algorithm

- First attempt:
  - Sort activities by finish time (after all we did this for the previous scheduling problem).
  - Start with $i = 1$. Then repeatedly choose next activity and assign to classroom i; next classroom is $i = i + 1 \bmod d$
- How do we prove this is correct?
  - This cannot be done. Beware of bad examples!
  - This “solution” works for the previous slide but in general it fails!

A Room Scheduling Algorithm

- We sort the activities by start time.
  - We still require the depth computation: the minimum number of rooms needed is d. We use d labels to name the rooms.
  - Assuming t
    respect to s
    as a_i.
  - Pseudo-cod

```
For $j$ = 1 to $n$
    For each $a_i$ that precedes $a_j$ and overlaps it
        Exclude the label of $a_i$ from the set of all labels
    Endfor
   If there is any label in the set of $d$ labels that has not
   been excluded then
       Assign a non-excluded label to $a_j$
   Else Leave $a_j$ unlabeled.
   Endif
Endfor
```

Proof for the Room Scheduling Algorithm (1)

- The following claims prove the success of this greedy algorithm:
- No activity goes unlabeled.
  - The "ELSE statement" will never execute.
  - Consider a_j and assume there are k activities prior to a_j in the sorted order that overlap a_j.
  - These k+1 activities *(a_j included)* form a set that must have a depth < d at the start time of a_j.
  - So, $k+1 < d$ implies $k < d$ -1 implies that at the start time of a_j there is a free label (i.e. a free room).

Proof for the Room Scheduling Algorithm (2)

- Also, no two overlapping activities can get the same label.
  - Suppose a_i and a_j overlap.
- Assume WLOG that a_i precedes a_j in the sorted order.
  - Then when a_j is considered by the algorithm, the label for a_i will be excluded and it cannot be assigned to a_j.
  - Notice how the proof has made use of the three main components of this greedy algorithm: the computation of the depth, the sorting, and the excluding operation.
- If you had a proof that avoided any of these components then something would be amiss: either the algorithm is suspect or the proof is wrong…

Scheduling to Minimize Lateness

- **Instance**: a set of n activities, each with start time s_i, deadline d_i and a duration t_i.
- **Problem**: we plan to satisfy each request, but we are allowed to let certain requests run late, and the optimization goal is to schedule all requests, using non
  to minimize the ma
  - We say a request i is
    and the lateness of s
    $l_i=f(i)-d_i$.
  - The goal: minimize $L=\max_i l_i$.

Greedy Strategies

- Choosing the smallest t_i
- Choosing the smallest (d_i-t_i)
- Choosing the smallest d_i

Greedy Scheduling Algorithm

- Sort task activities by deadline.
    - Let time stamp $Q = 0$
- Repeatedly assign to task $i = 1..n$
    - $s(i) = Q$
    - $Q = Q + t_i$
    - $f(i) = Q$
- Note that the t_i are used in the Q computations but not in the determination of job order. (!)
- How do we prove this greedy schedule gives an optimal solution?

Proof of the Correctness

- There is an optimal schedule with no idle time.
- All schedules with no inversions and no idle time have the same maximum lateness.
  - Inversion: if a job *i* with deadline d_i is scheduled before another job j with deadline $d_j < d_i$.
- There is an optimal schedule that has no inversions and no idle time.