4-9 Linear Code

(From the Perspective of Linear Algebra)

Hengfeng Wei

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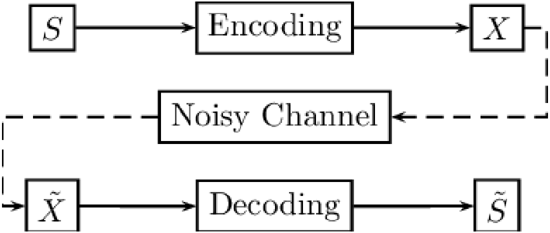
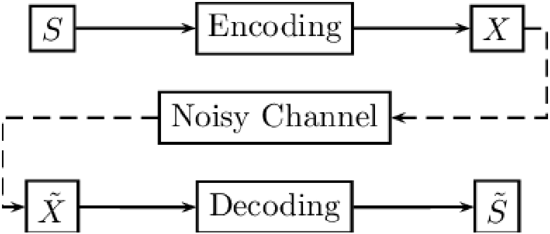
May 13, 2019


Welcome to
Linear
Algebra


S
Encoding
X
Noisy Channel
$\tilde{X}$
Decoding
$\tilde{S}$



Q : Where is Cryptography?

$$\boxed{\mathrm{Col}(G_{n\times k}) = C = \mathrm{Nul}(H_{(n-k)\times n})}$$

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$$(n, k, d)$$

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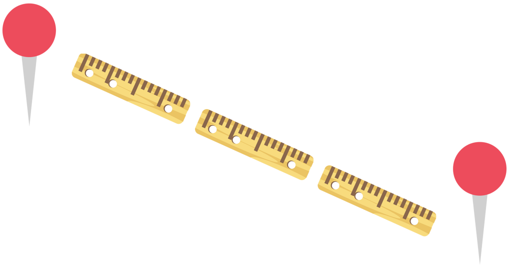
n : length

k : # of information bits

d : distance

$\text{Hamming}(7, 4, 3)$



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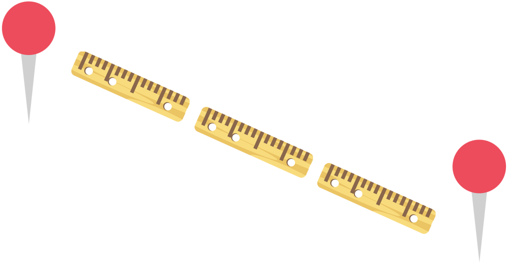

Detect $d - 1$ errors

Correct $\lfloor \frac{d-1}{2} \rfloor$ errors

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Definition (Linear Code)

A linear code C of length n is a linear subspace of the vector space $\mathbb{Z}_2^n$ ($\mathbb{F}_q^n$).

$$c_1 \in C, c_2 \in C \implies c_1 + c_2 \in C$$

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$$\begin{aligned} d(C) &= \min\left\{d(c_1, c_2) \mid c_1 \neq c_2, c_1, c_2 \in C\right\} \\ &= \min\left\{w(c_1 + c_2) \mid c_1 \neq c_2, c_1, c_2 \in C\right\} \\ &= \min\left\{w(c) \mid c \neq 0, c \in C\right\} \end{aligned}$$

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Show that either every codeword has even weight or exactly half of them have even weight.

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$$C = \text{Span}(c_1, c_2, \cdots, c_k)$$

Definition (Generator Matrix)

A matrix $G_{n \times k}$ is a generator matrix for an (n, k) linear code C if

$$C = \text{Col}(G)$$

$$\boxed{G_{n \times k} = \begin{bmatrix} c_1 & c_2 & \cdots & c_k \end{bmatrix}}$$

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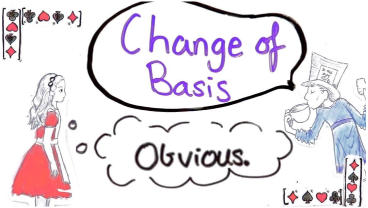

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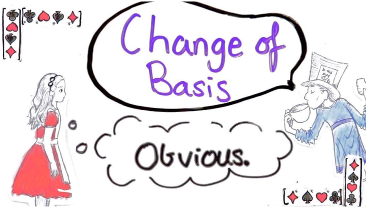
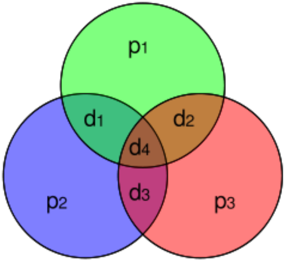


Definition (Standard Generator Matrix)

$$G_{n\times k} = \begin{bmatrix} I_k \\ A_{(n-k)\times k} \end{bmatrix}$$

Generator matrix for Hamming code $(7, 4, 3)$



$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

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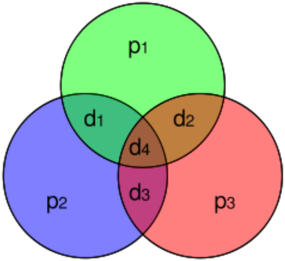

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$$G \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$G \cdot \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{bmatrix} \color{red}{1} & 0 & 0 & 0 \\ 0 & \color{red}{1} & 0 & 0 \\ 0 & 0 & \color{red}{1} & 0 \\ 0 & 0 & 0 & \color{red}{1} \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

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Each parity-check bit is a linear combination of some data bits.

$$
\begin{aligned}
d_1 + d_2 \qquad + d_4 + p_1 \qquad\qquad &= 0 \\
d_2 + d_3 + d_4 \qquad + p_2 \qquad &= 0 \\
d_1 \qquad + d_3 + d_4 \qquad\qquad + p_3 &= 0
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$$\begin{aligned}
H_{(n-k)\times n} \cdot G_{n\times k} & \\
&= \left[A_{(n-k)\times k} \mid I_{n-k} \right] \cdot \begin{bmatrix} I_k \\ A_{(n-k)\times k} \end{bmatrix} \\
&= A_{(n-k)\times k} \cdot I_k + I_{n-k} \cdot A_{(n-k)\times k} \\
&= A_{(n-k)\times k} + A_{(n-k)\times k} \\
&= 0_{(n-k)\times k}
\end{aligned}$$

$$r = c + e_i$$

$$r = c + (e_i + e_j + \cdots)$$

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Definition (Syndrome)

$$\begin{aligned} S(r) &= Hr \\ &= H(c + (e_i + e_j + \cdots)) \\ &= H(e_i + e_j + \cdots) \\ &= He_i + He_j + \cdots \end{aligned}$$

Theorem (Extracting $d(C)$ from H)

If H is the parity-check matrix for a linear code C, then $d(C)$ equals the minimum number of linearly dependent columns of H.

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$$\sum_{i=1}^{n} (c_i \cdot H_i) = 0$$

H_i : the i^{th} column of H

□

Theorem (Single Error-detecting Code (Theorem 8.31))

$$\begin{aligned} d(C) \geq 2 & \\ & \iff \forall \{c_i\} \ \textit{linearly independent} \\ & \iff \textit{no zero column} \end{aligned}$$

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\end{aligned}
$$

Theorem (Single Error-correcting Code (Theorem 8.34))

$$
\begin{aligned}
d(C) &\geq 3 \\
&\iff \forall \{c_i, c_j\} \text{ linearly independent} \\
&\iff \text{no zero column, no identical columns}
\end{aligned}
$$

Problem 8.5-21

If we are to use an error-correcting linear code to transmit the 128 ASCII characters, what size matrix must be used?

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$$r \triangleq n - k \quad (k = 7)$$

$$k \le 2^r - 1 - r \implies r \ge 4$$

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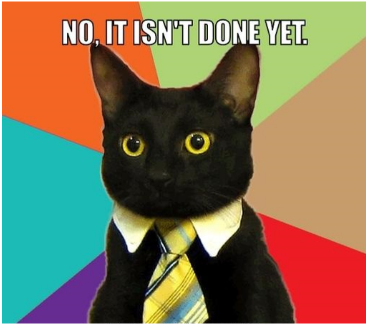
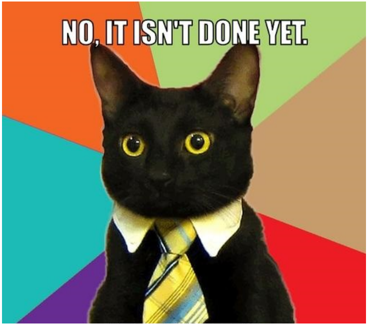
$$H_{4\times 11}: \quad (11, 7) \text{ code}$$


NO, IT ISN'T DONE YET.



Hamming Code (wiki):
General Algorithm

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$$r \triangleq n - k = 1 \text{ is sufficient} : (8, 7) \text{ code}$$

Problem 8.5-23

How many check positions are needed for a single error-correcting code with $k = 20$?

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$$r \triangleq n - k \quad (k = 20)$$

$$k \leq 2^r - 1 - r \implies r \geq 5$$

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$$H_{(n-k)\times n} = H_{1\times 4} = [1, 1, 1, 1] \qquad G_{n\times k} = G_{4\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

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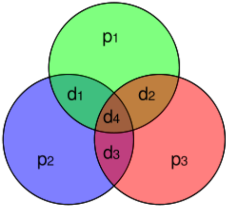

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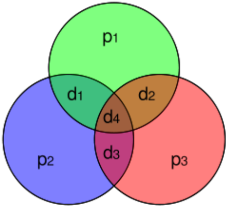

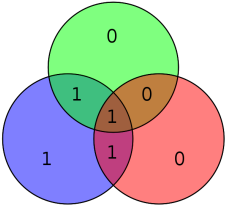

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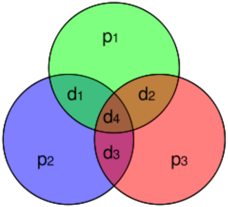

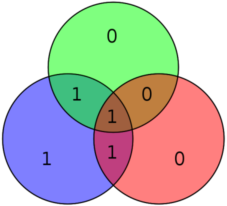

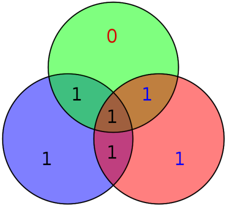

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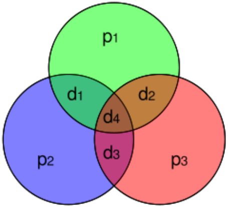
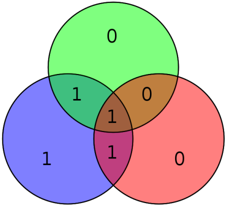
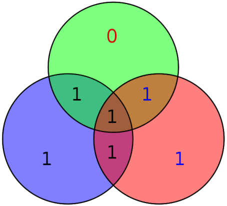
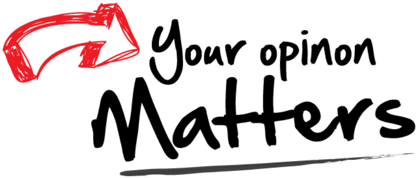






Hamming$(7, 4, 3)$ cannot distinguish between single-bit errors and two-bit errors.


THANK YOU



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