Red-Black Trees

A little history

- 1962: The idea of balancing a search tree is due to Adel'son-Velskii and Landis.
- 1970: Hopcroft introduced 2-3 trees. (B-tree is a generalization of it)
- 1972: Bayer invented Red-black trees.
- 1978: Guibas and Sedgewick introduced the red/black convention.

Red-Black Properties

- The *red-black properties*:
  1. Every node is either red or black
  2. The root is always black
  3. Every leaf (NULL pointer) is black
     - Note: this means every “real” node has 2 children
  4. If a node is red, both children are black
     - Note: can’t have 2 consecutive reds on a path
  5. Every path from node to descendent leaf contains the same number of black nodes

Black-Height

- *black-height*: # black nodes on path to leaf
- *What is the minimum black-height of a node with height h?*
- A: a height-h node has black-height $\geq h/2$
- Theorem: A red-black tree with n internal nodes has height $h \leq 2 \lg(n + 1)$
  - Proved by induction

RB Trees: Proving Height Bound

- Prove: n-node RB tree has height $h \leq 2 \lg(n+1)$
- Claim: A subtree rooted at a node x contains at least $2^{\text{bh}(x)} - 1$ internal nodes
  - Proof by induction on height h
  - Base step: x has height 0 (i.e., NULL leaf node)
    - *What is bh(x)?*

RB Trees: Proving Height Bound

- Prove: n-node RB tree has height $h \leq 2 \lg(n+1)$
- Claim: A subtree rooted at a node x contains at least $2^{bh(x)} - 1$ internal nodes
  - Proof by induction on height h
  - Base step: x has height 0 (i.e., NULL leaf node)
    - *What is bh(x)?*
    - A: 0
    - So…subtree contains $2^{bh(x)} - 1$
      $= 2^0 - 1$
      $= 0$ internal nodes (TRUE)

RB Trees: Proving Height Bound

- Inductive proof that subtree at node x contains at least $2^{\text{bh}(x)} - 1$ internal nodes
  - Inductive step: x has positive height and 2 children
    - Each child has black-height of $\text{bh}(x)$ or $\text{bh}(x)-1$ *(Why?)*
    - The height of a child = (height of x) - 1
    - So the subtrees rooted at each child contain at least $2^{\text{bh}(x) - 1} - 1$ internal nodes
    - Thus subtree at x contains
      $(2^{\text{bh}(x) - 1} - 1) + (2^{\text{bh}(x) - 1} - 1) + 1$
      $= 2 \bullet 2^{\text{bh}(x)-1} - 1 = 2^{\text{bh}(x)} - 1$ nodes

Proving Height Bound

- Thus at the root of the red-black tree:

  $n \geq 2^{\mathrm{bh}(root)} - 1$

  $n \geq 2^{h/2} - 1$

  $\lg(n+1) \geq h/2$

  $h \leq 2\lg(n+1)$

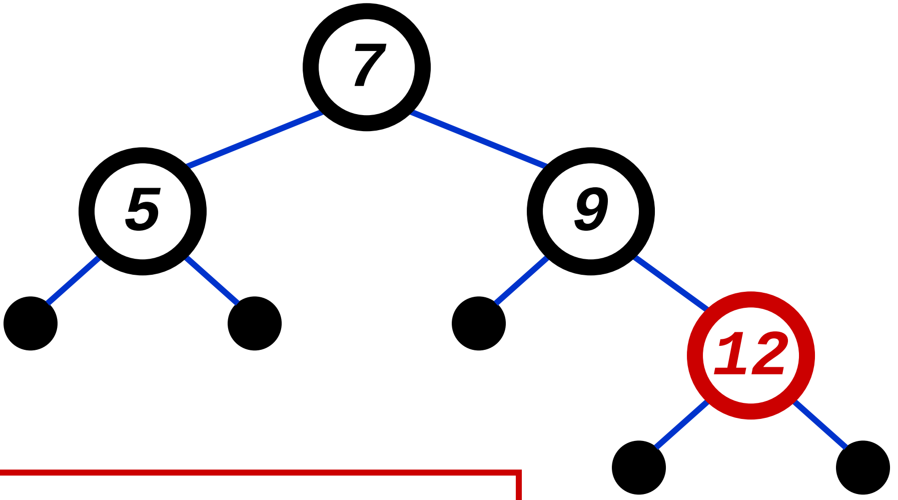
  Thus $h = O(\lg n)$

RB Trees: Worst-Case Time

- So we've proved that a red-black tree has O(lg *n*) height
- Corollary: These operations take O(lg *n*) time:
  - Minimum(), Maximum()
  - Successor(), Predecessor()
  - Search()
- Insert() and Delete():
  - Will also take O(lg *n*) time
  - But will need special care since they modify tree

Red-Black Trees: An Example

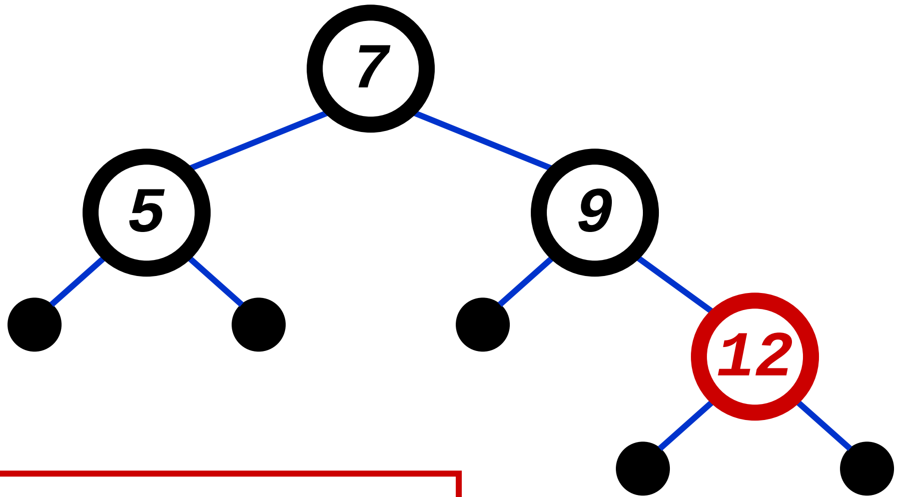
- *Color this tree:*



Red-black properties:
1. Every node is either red or black
2. The root is always black
3. Every leaf (NULL pointer) is black
4. If a node is red, both children are black
5. Every path from node to descendent leaf
   contains the same number of black nodes

Red-Black Trees: The Problem With Insertion

- Insert 8
  - *Where does it go?*



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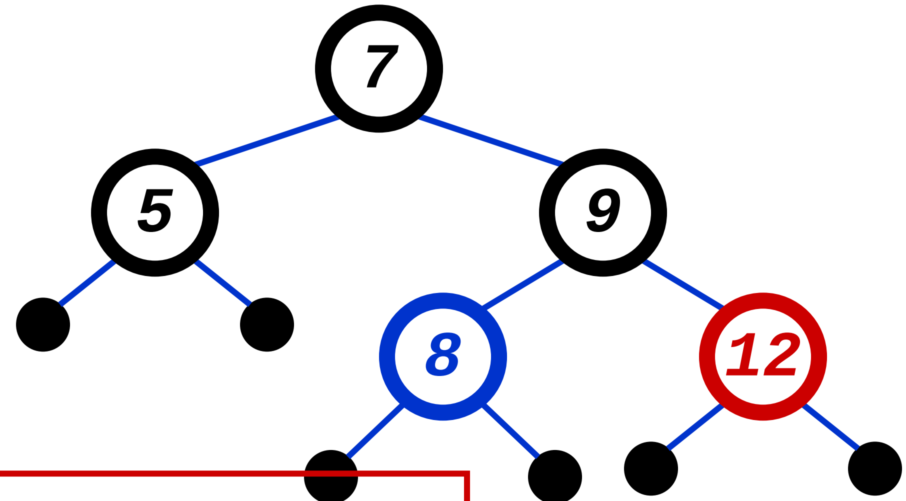
Red-Black Trees: The Problem With Insertion

- Insert 8
  - *Where does it go?*
  - *What color should it be?*



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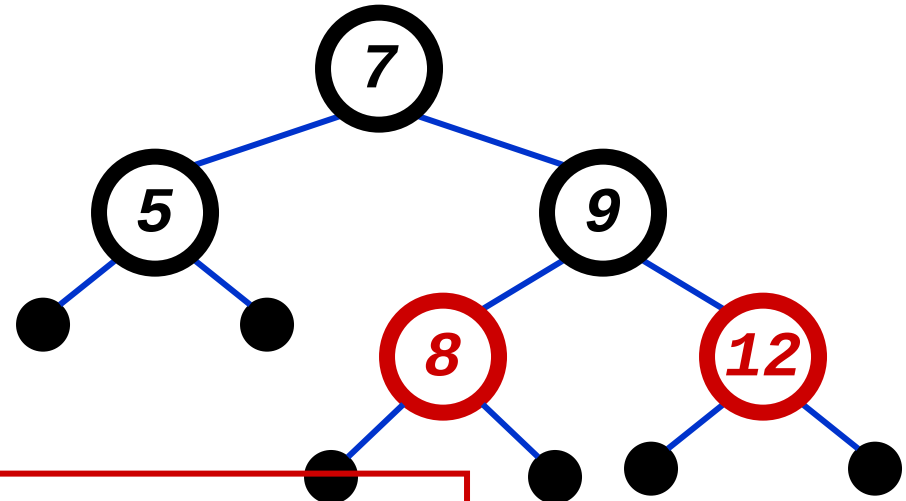

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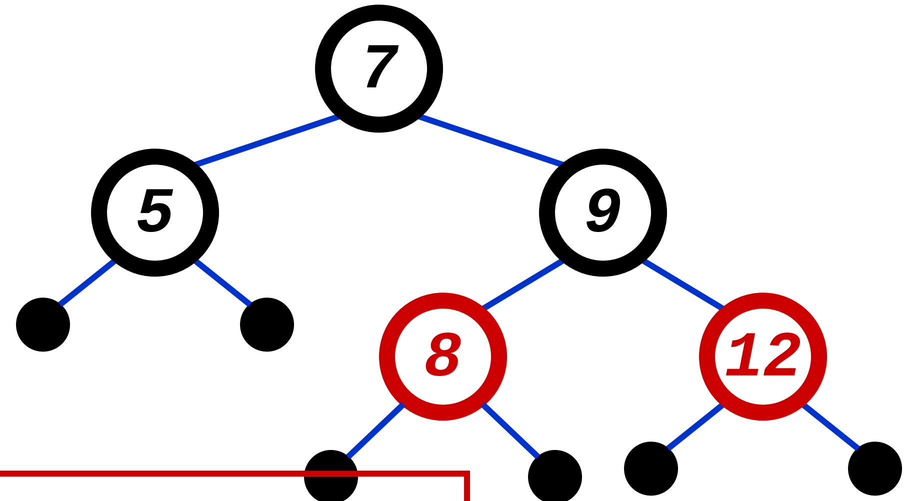
Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*



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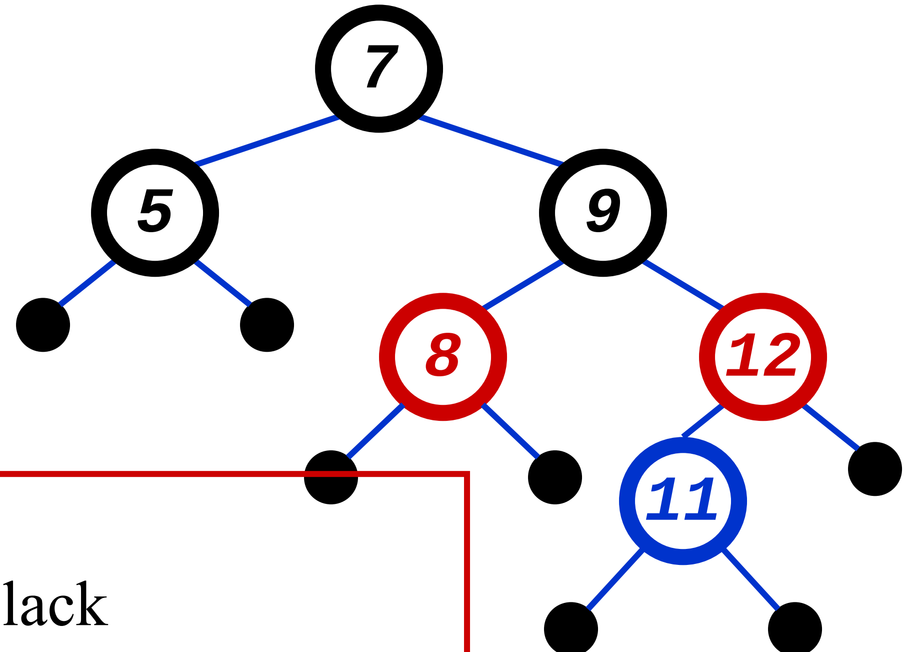
  - *Where does it go?*
  - *What color?*



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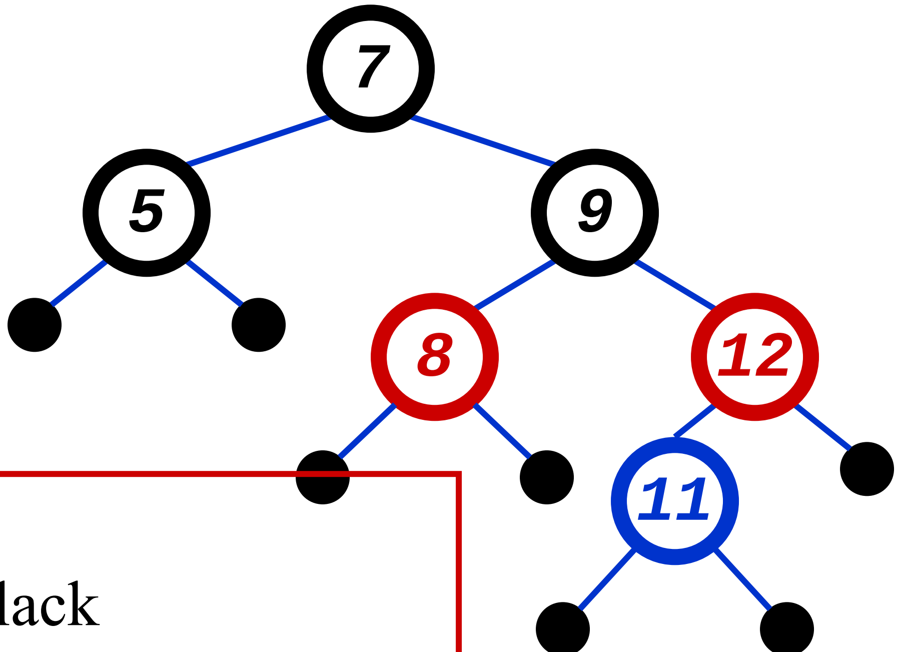
Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*
  - *What color?*
    - Can't be red! (#4)



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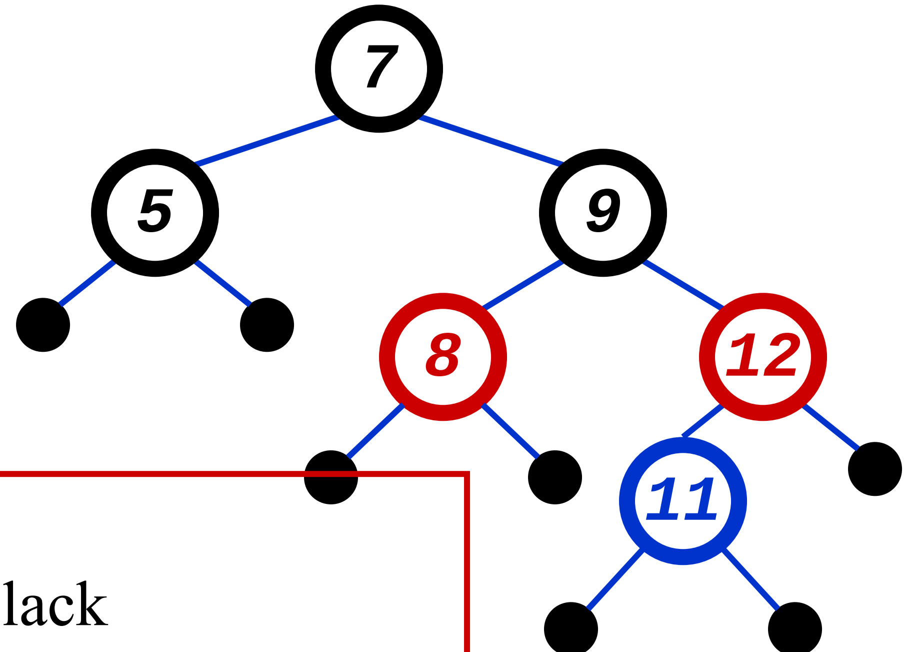
Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*
  - *What color?*
    - Can't be red! (#4)
    - Can't be black! (#5)



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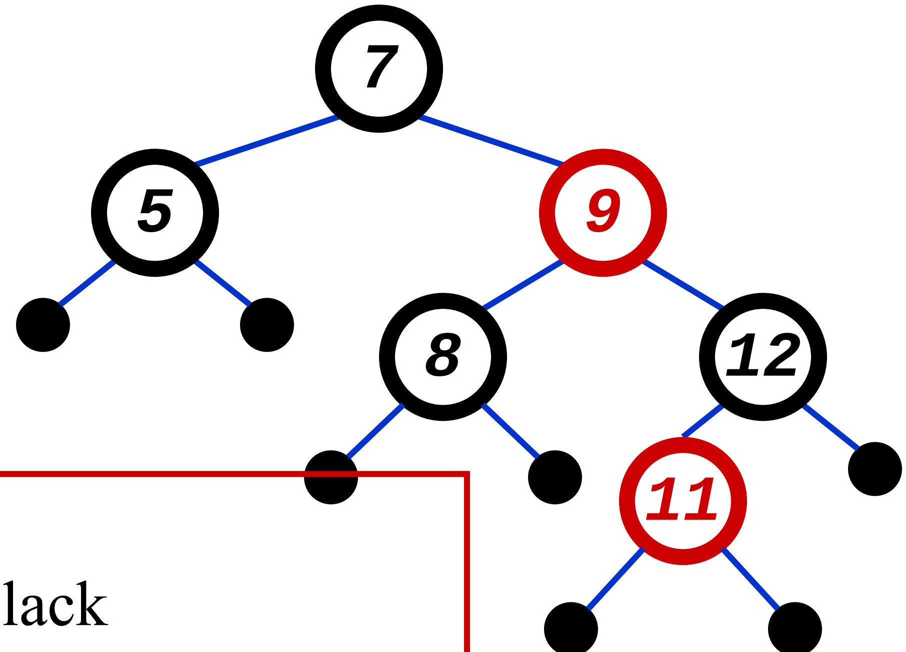
Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*
  - *What color?*
    - Solution: recolor the tree



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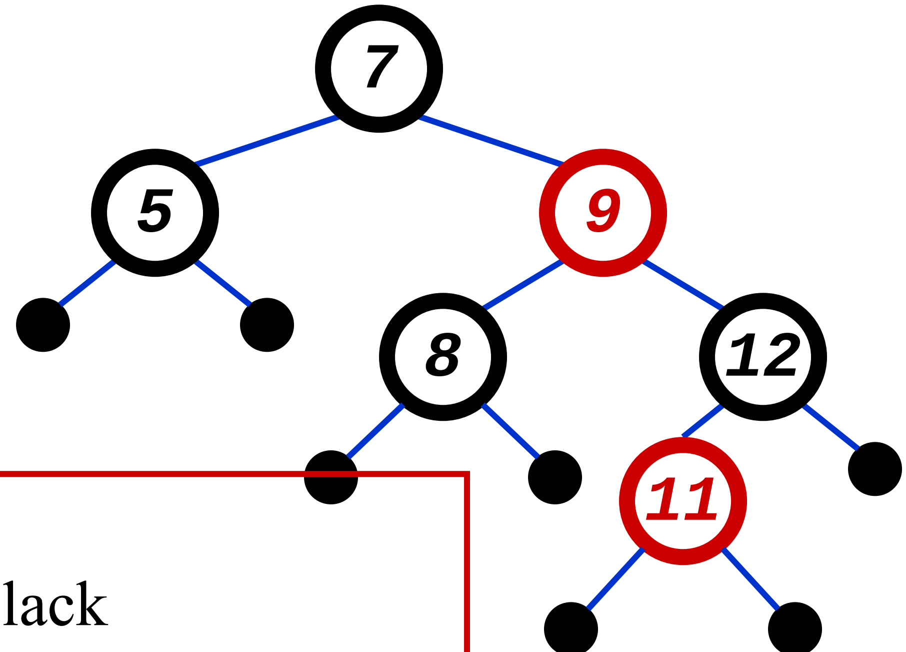
Red-Black Trees: The Problem With Insertion

- Insert 10
  - *Where does it go?*



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Red-Black Trees: The Problem With Insertion

- Insert 10
  - *Where does it go?*
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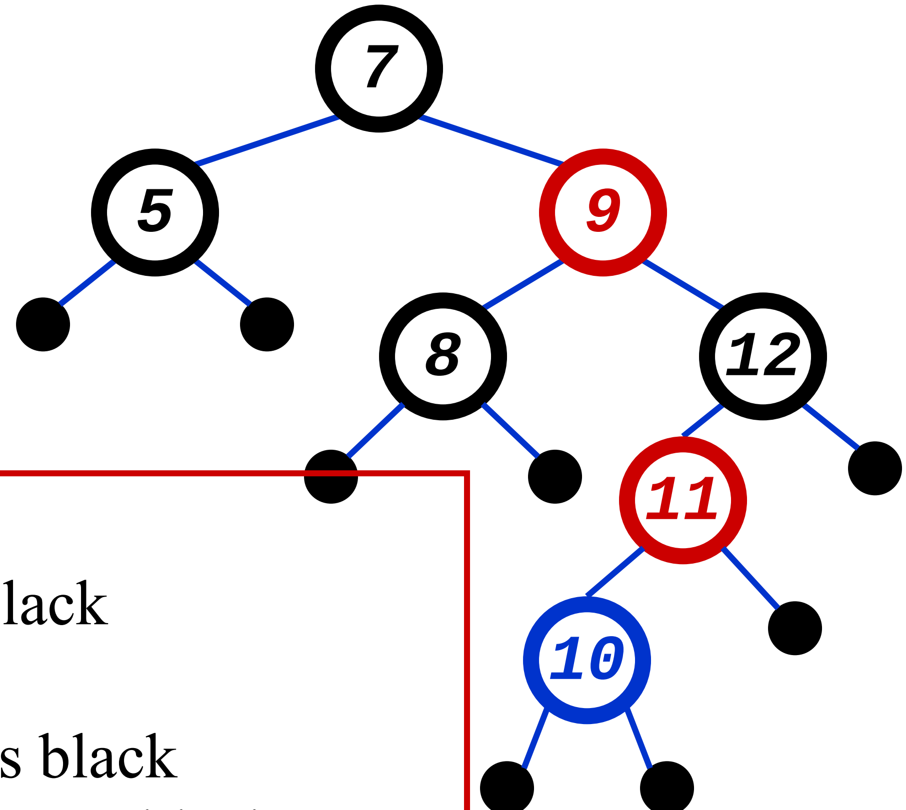

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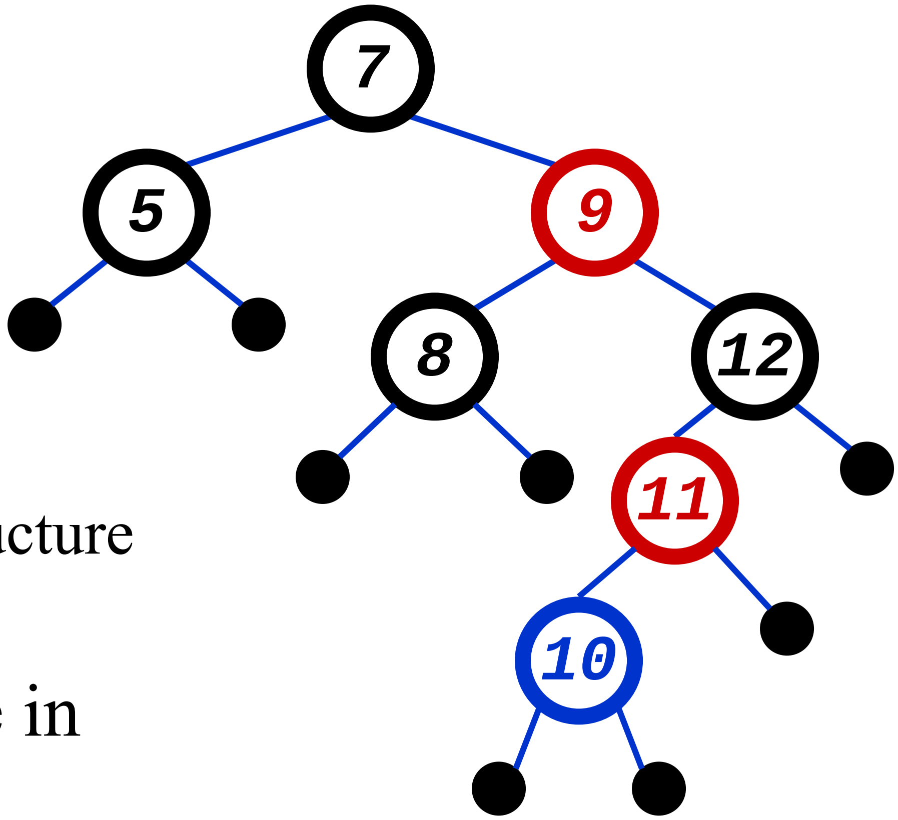
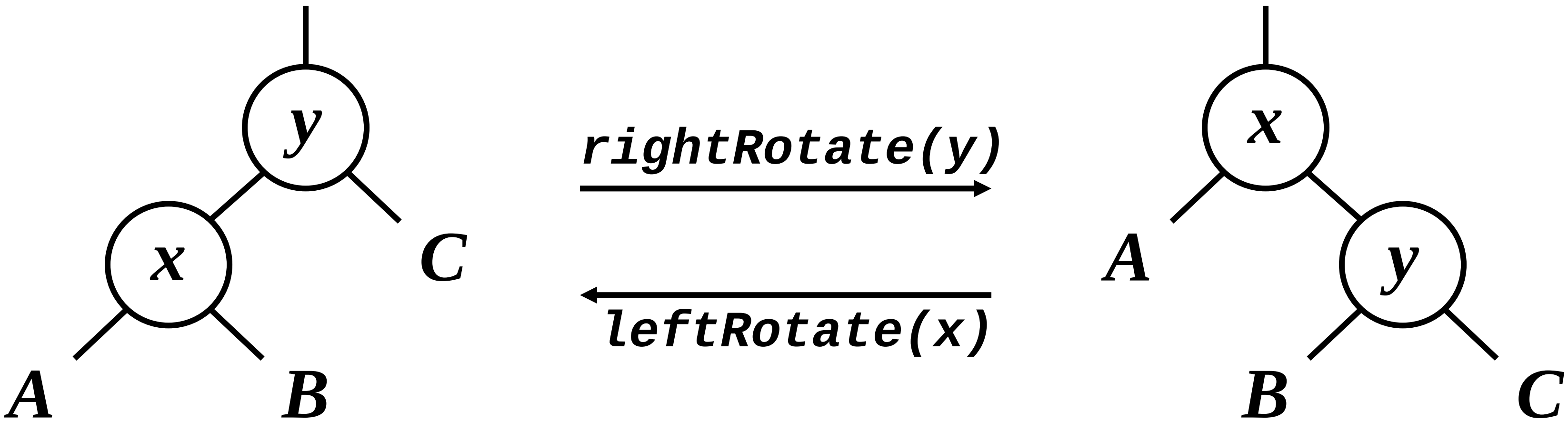
Red-Black Trees: The Problem With Insertion

- Insert 10
  - *Where does it go?*
  - *What color?*
    - A: no color! Tree is too imbalanced
    - Must change tree structure to allow recoloring
  - Goal: restructure tree in O(lg *n*) time



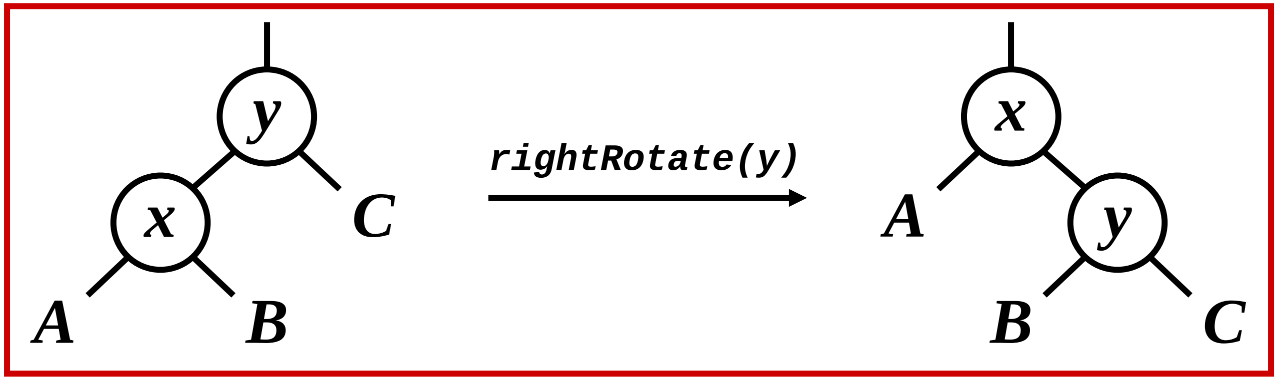
RB Trees: Rotation

- Our basic operation for changing tree structure is called *rotation*:



- *Does rotation preserve inorder key ordering?*
- *What would the code for* ***`rightRotate()`*** *actually do?*

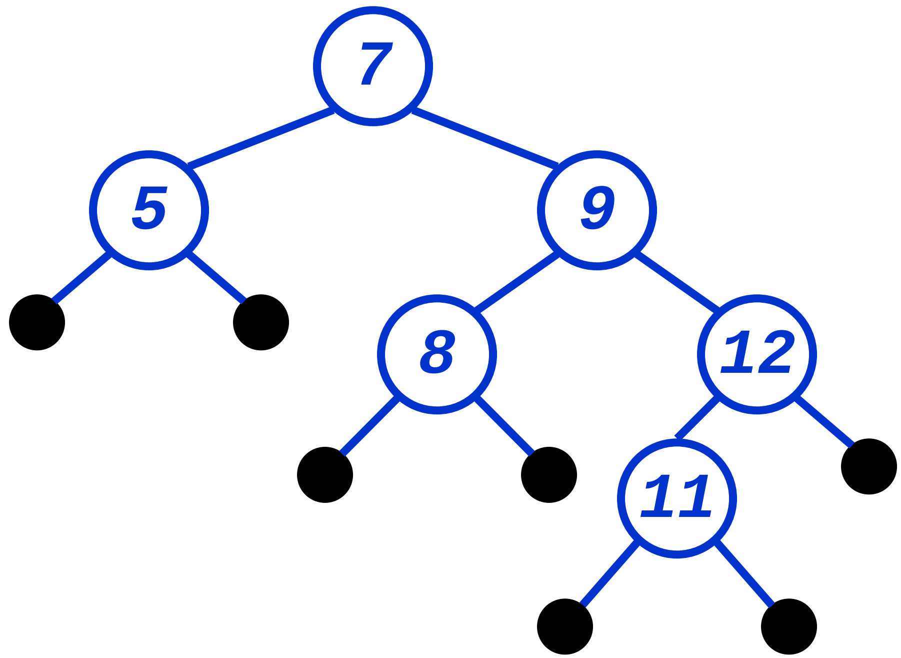
RB Trees: Rotation



- Answer: A lot of pointer manipulation
  - *x* keeps its left child
  - *y* keeps its right child
  - *x*'s right child becomes *y*'s left child
  - *x*'s and *y*'s parents change
- *What is the running time?*

Rotation Example

- Rotate left about 9:



Rotation Example

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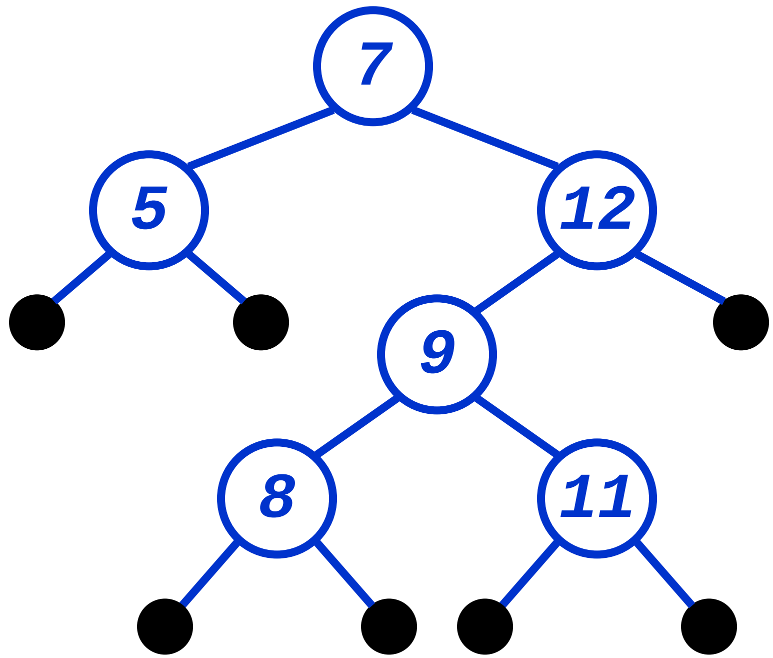
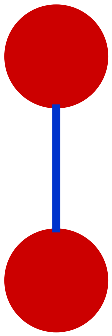


Red-Black Trees: Insertion

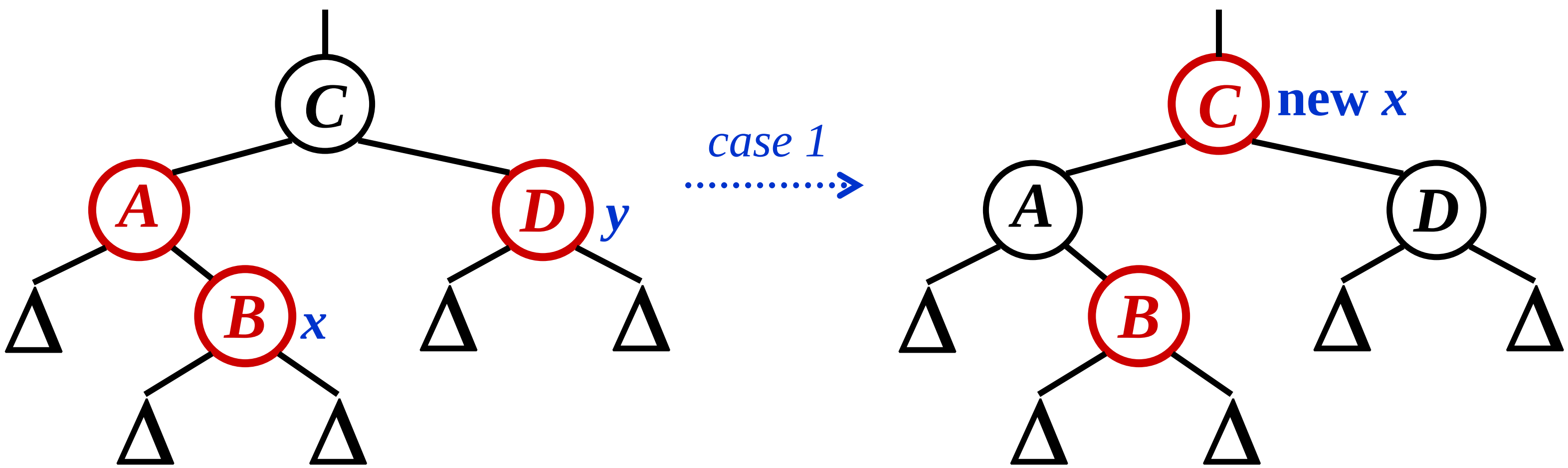
- Insertion: the basic idea
  - Insert *x* into tree, color *x* red
  - Only r-b property 4 might be violated (if p[*x*] red)
    - If so, move violation up tree until a place is found where it can be fixed
  - Total time will be O(lg *n*)



RB Insert: Case 1

```
if (y->color == RED)
    x->p->color = BLACK;
    y->color = BLACK;
    x->p->p->color = RED;
    x = x->p->p;
```

- Case 1: “uncle” is red
- In figures below, all Δ’s are equal-black-height subtrees



Change colors of some nodes, preserving #5: all downward paths have equal b.h.
The while loop now continues with x’s grandparent as the new x

RB Insert: Case 1

```
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    y->color = BLACK;
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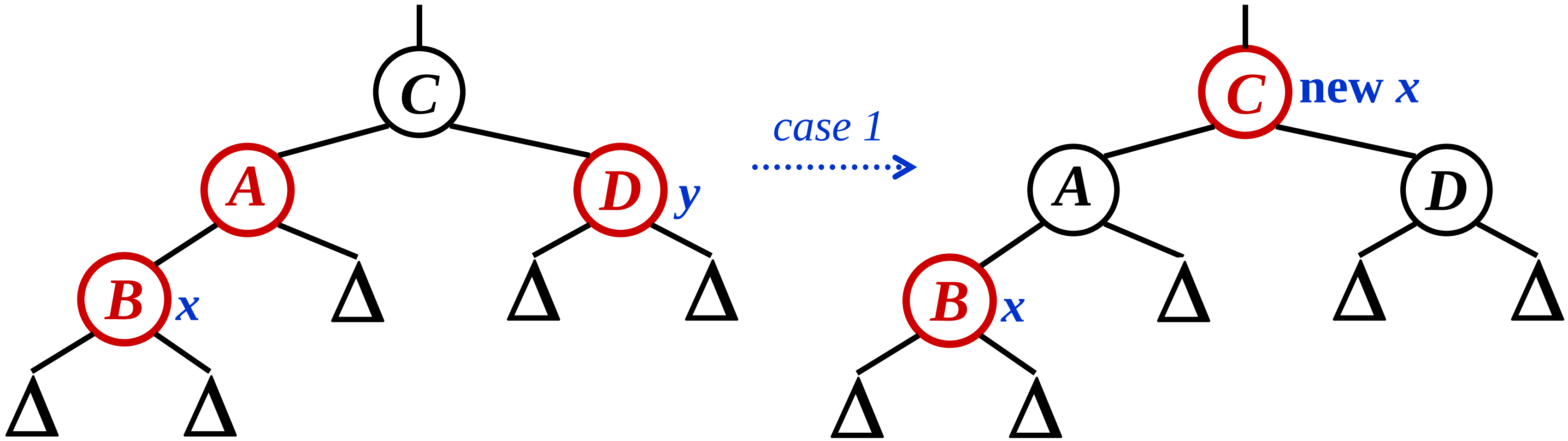
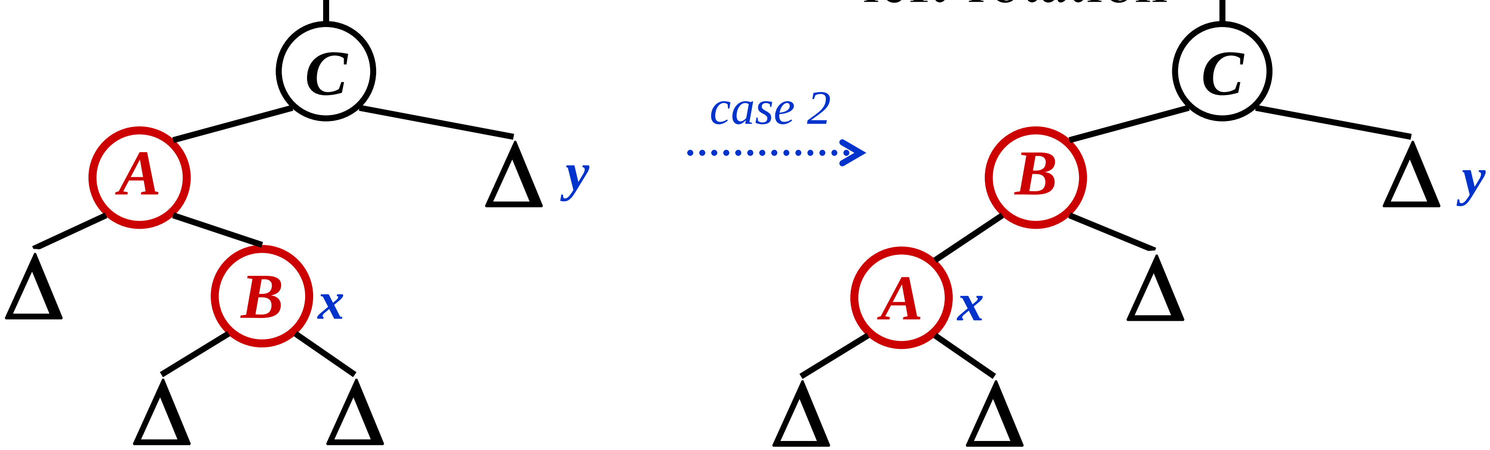


Same action whether x is a left or a right child

RB Insert: Case 2

```
if (x == x->p->right)
    x = x->p;
    leftRotate(x);
// continue with case 3 code
```

- Case 2:
  - “Uncle” is black
  - Node *x* is a right child
- Transform to case 3 via a left-rotation

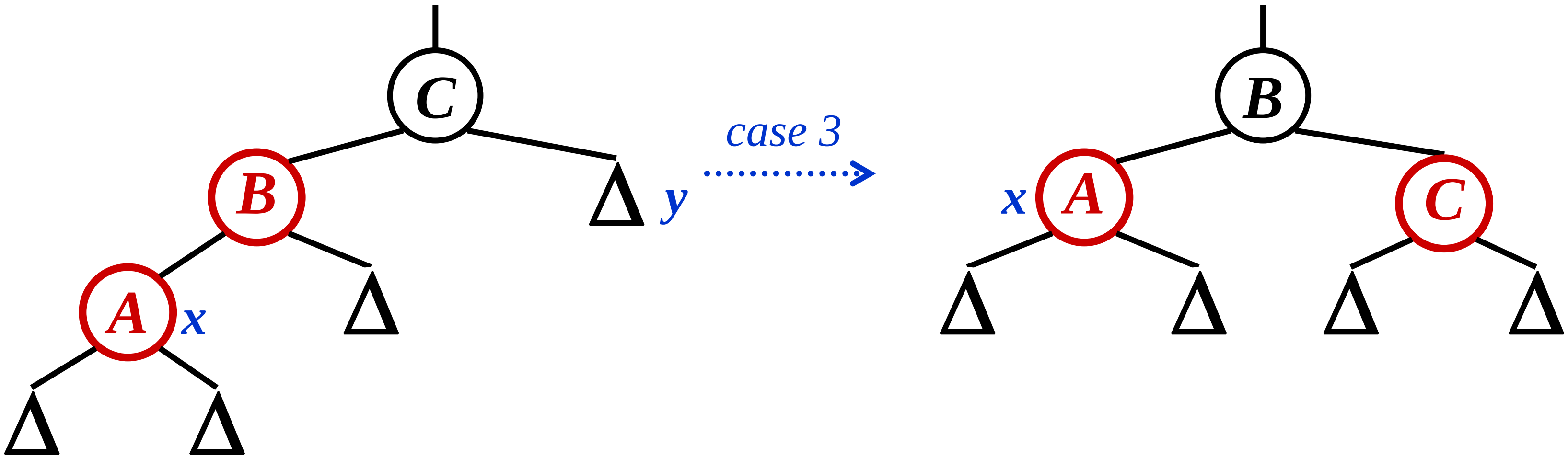


Transform case 2 into case 3 (x is left child) with a left rotation
This preserves property 5: all downward paths contain same number of black nodes

RB Insert: Case 3

```
x->p->color = BLACK;
x->p->p->color = RED;
rightRotate(x->p->p);
```

- Case 3:
  - "Uncle" is black
  - Node *x* is a left child
- Change colors; rotate right



Perform some color changes and do a right rotation

Again, preserves property 5: all downward paths contain same number of black nodes

RB Insert: Cases 4-6

- Cases 1-3 hold if *x*'s parent is a left child
- If *x*'s parent is a right child, cases 4-6 are symmetric (swap left for right)

```
rbInsert(x)
  treeInsert(x);
  x->color = RED;
  // Move violation of #4 up tree, maintaining #5 as invariant:
  while (x!=root && x->p->color == RED)
  if (x->p == x->p->p->left)
      y = x->p->p->right;
      if (y->color == RED)
          x->p->color = BLACK;
          y->color = BLACK;                 Case 1
          x->p->p->color = RED;
          x = x->p->p;
      else    // y->color == BLACK
          if (x == x->p->right)
              x = x->p;                     Case 2
              leftRotate(x);
          x->p->color = BLACK;
          x->p->p->color = RED;             Case 3
          rightRotate(x->p->p);
  else    // x->p == x->p->p->right
      (same as above, but with
       "right" & "left" exchanged)
```

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  if (x->p == x->p->p->left)
      y = x->p->p->right;
      if (y->color == RED)
          x->p->color = BLACK;
          y->color = BLACK;                 Case 1: uncle is RED
          x->p->p->color = RED;
          x = x->p->p;
      else   // y->color == BLACK
          if (x == x->p->right)
              x = x->p;                     Case 2
              leftRotate(x);
          x->p->color = BLACK;
          x->p->p->color = RED;             Case 3
          rightRotate(x->p->p);
  else    // x->p == x->p->p->right
      (same as above, but with
       “right” & “left” exchanged)
```

Red-Black Trees: Deletion

- And you thought insertion was tricky…

Red-Black Trees

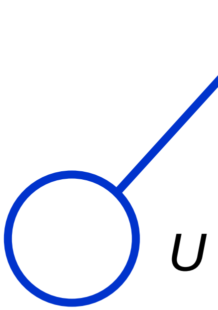
Bottom-Up Deletion

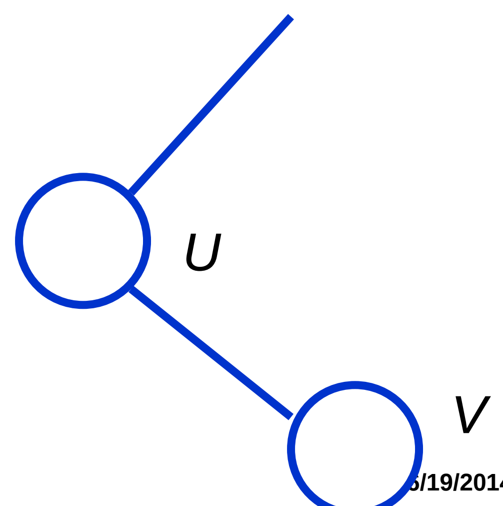
Recall “ordinary” BST Delete

1. If vertex to be deleted is a leaf, just delete it.
2. If vertex to be deleted has just one child, replace it with that child
3. Otherwise, if vertex Z has both a left and a right child. We find Z’s successor U, replace Z’s value by U’s value and then delete U (a recursive step, and U must be a leaf or has just one child).

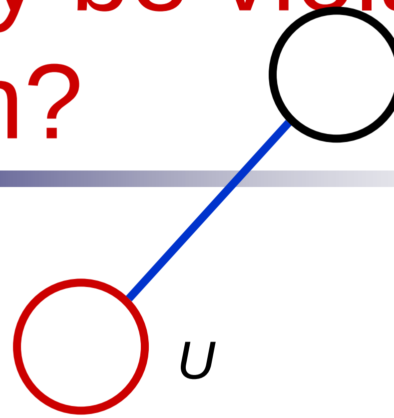
Bottom-Up Deletion

1. Do ordinary BST deletion. Eventually a “case 1” or “case 2“ will be done (leaf or just one child). If deleted node, U, is a leaf, think of deletion as replacing with the NULL pointer, V. If U had one child, V, think of deletion as replacing U with V.
2. What can go wrong??





Which RB Property may be violated after deletion?



1. If U is red?

   Not a problem – no RB properties violated

2. If U is black?

   If U is not the root, deleting it will change the black-height along some path

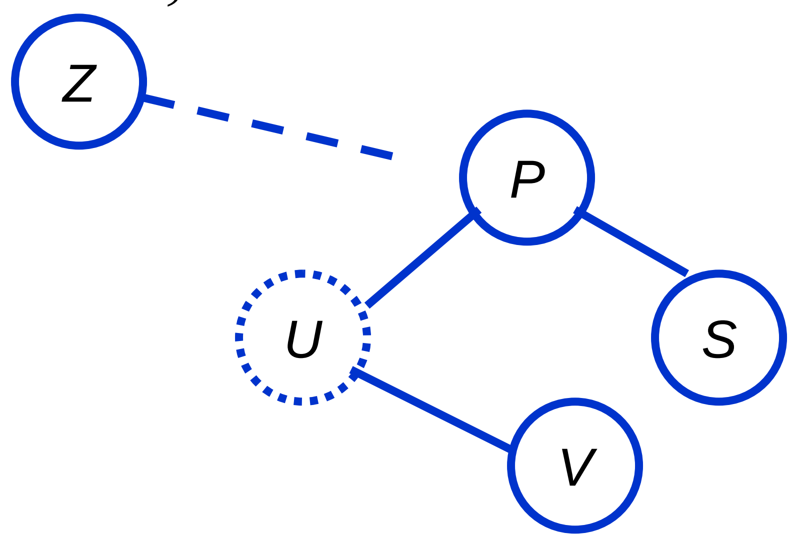
Fixing the problem

- Think of V (NULL pointer or U’s only child) as having an “extra” unit of blackness. This extra blackness must be absorbed into the tree (by a red node), or propagated up to the root (without violating the RB properties) and out of the tree.

- If V is red, then we color it black to make it absorb the extra black. Otherwise, V is “double black”.
- There are four cases – our examples and “rules” assume that V is a left child. There are symmetric cases for V as a right child

Terminology

- The node just deleted was U (Z' successor!)
- The node that replaces it is V, which has an extra unit of blackness
- The parent of V is P
- The sibling of V is S



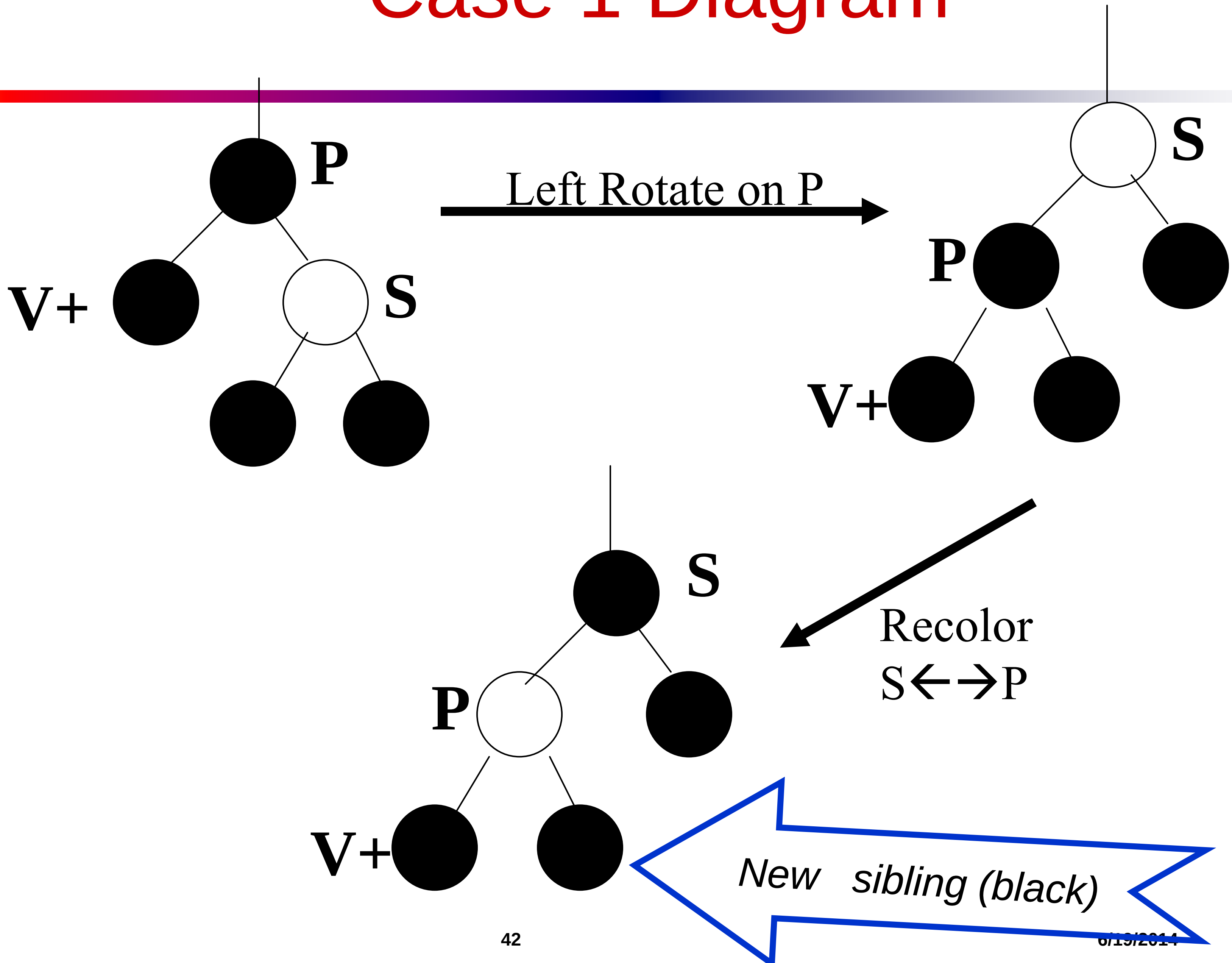
Black Node

Red Node

Red or Black and don't care

- 4 cases:
  - Case 1: V's sibling S is red; → Case 2/3/4
  - Case 2: V's sibling S is black; S's both children are black; →recursive or terminal
  - Case 3: V's sibling S is black; S's left child is red; S's right child is black; → Case 4
  - Case 4: V's sibling S is black; S's left child is red/black; S's right child is red; terminal case

Case 1 Diagram



Bottom-Up Deletion Case 1

- V's sibling, S, is Red
  - Left Rotation on P and recolor S & P
- NOT a terminal case – One of the other cases will now apply
- All other cases apply when S is Black

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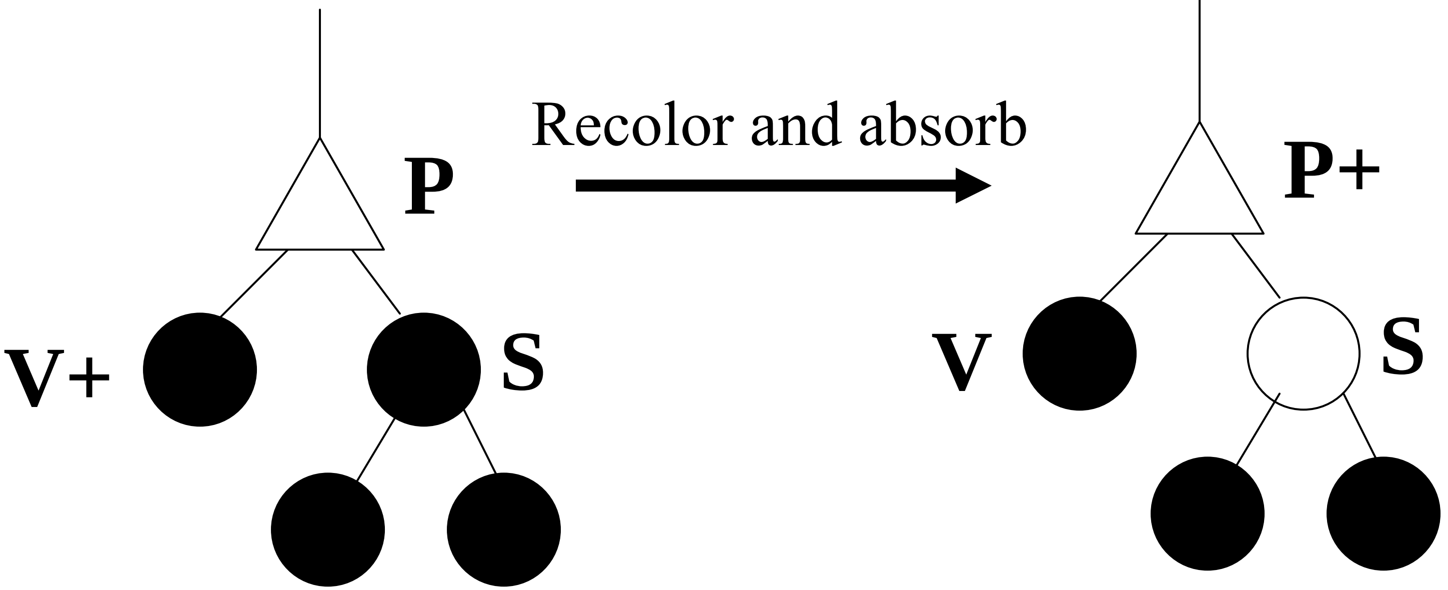
Case 2 diagram



Either extra black absorbed by P (P was Red, now case done) or P now has extra blackness (P was black, now recursive at P+.)

Bottom-Up Deletion Case 2

- V's sibling, S, is black and has <u>two black children</u>.
  - Recolor S to be Red
  - P absorbs V's extra blackness
    - If P is Red, we're done
    - If P is Black, it now has extra blackness and problem has been propagated up the tree

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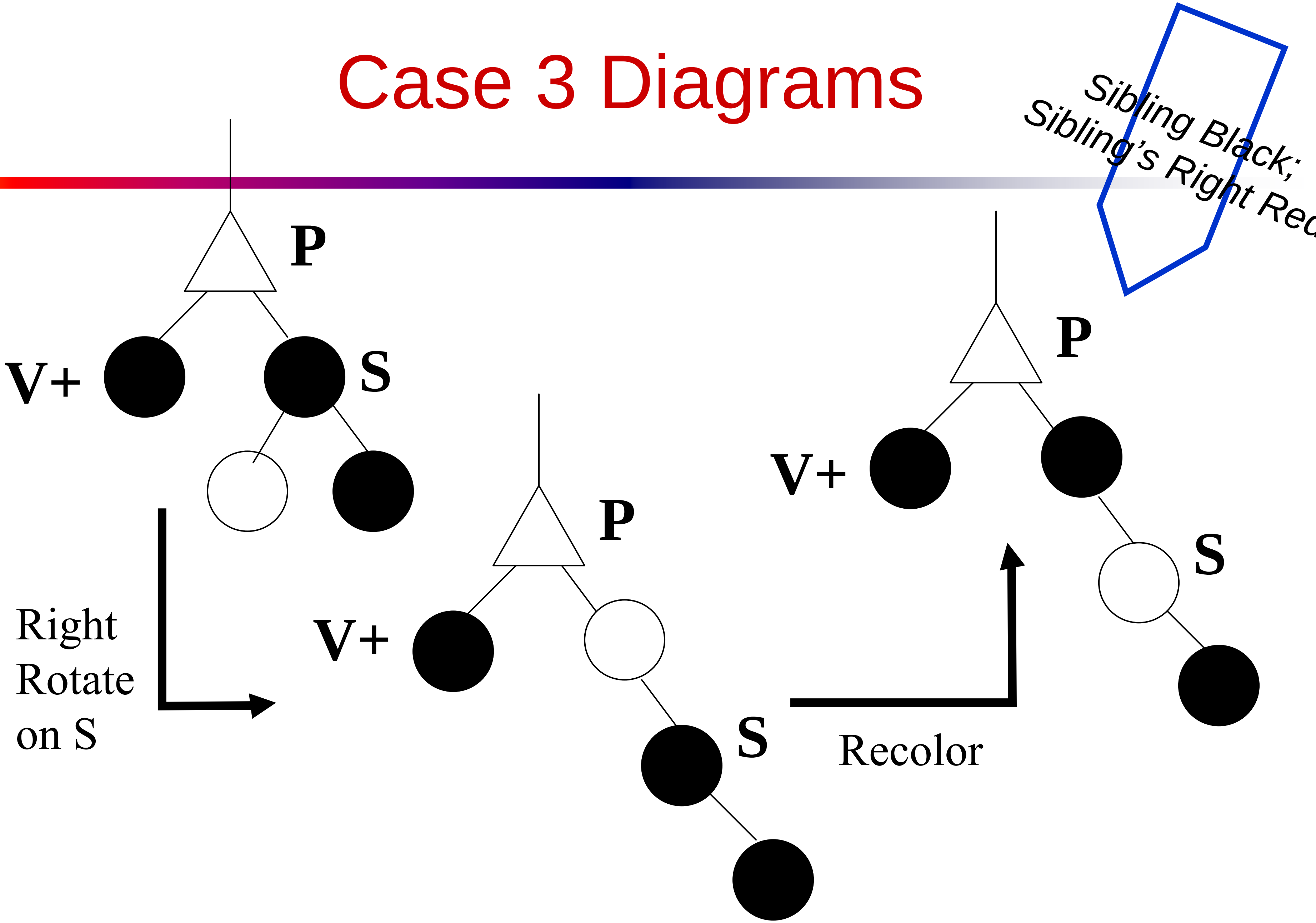
Case 3 Diagrams

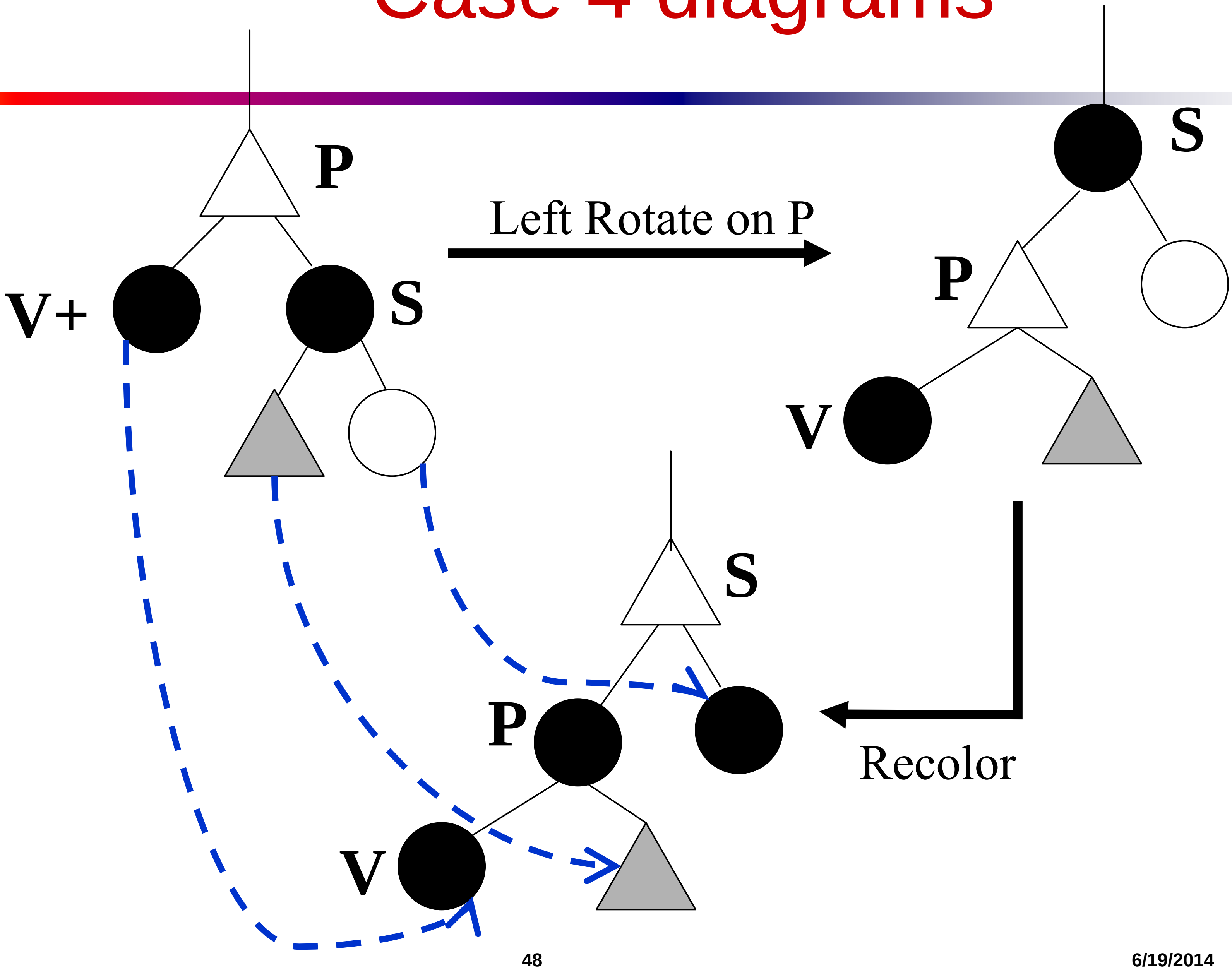


Bottom-Up Deletion
Case 3

- S is Black, S's right child is Black and S's left child is Red
  - Right Rotate on S
  - Swap color of S and S's left child
  - Now in case 4

Back to Case Map

Case 4 diagrams



Bottom-Up Deletion
Case 4

- S is black
- S's RIGHT child is RED (Left child either color)
  - Rotate S around P
  - Swap colors of S and P, and color S's Right child Black

- This is the terminal case – we're done

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The End