作业1-10

UD第13章问题**3**、4、**5**、11、13

UD第14章问题**8**、12、**13**、15

UD第15章问题1、6、7、**11**、12、13、14、**15**、20

UD第16章问题19、20、**21**、22

UD第27章项目6

Problem 13.3 Which of the following are functions from the set A to the set B? Give reasons for your answers.

(g) Define $f : \mathbb{Q} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x+1 & \text{if } x \in 2\mathbb{Z} \\ x-1 & \text{if } x \in 3\mathbb{Z} \\ 2 & \text{otherwise} \end{cases} .$$

f(6)?

Problem 13.5. Let X be a nonempty set and let A be a subset of X. The **characteristic function** or **indicator function** of the set A in X is

$$\chi_A : X \to \{0,1\} \text{ defined by } \chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \in X \setminus A \end{cases}.$$

(a) Since this is called the characteristic function, it probably is a function, but check this carefully anyway.

(b) Determine the domain and range of this function. Make sure you look at all possibilities for A and X.

$$Dom(f) = X$$
$$ran(f) = \{0,1\}$$

The domain of the function is the set X.

If $A = \varnothing$, then $ran(f) = \{0\}$. If $A \neq \varnothing$ and $A \subset X$, then $ran(f) = \{0, 1\}$. If $A = X$, then $ran(f) = \{1\}$.

Problem 14.8. For each of the functions below, determine whether or not the function is one-to-one and whether or not the function is onto. If the function is not one-to-one, give an explicit example to show what goes wrong. If it is not onto, determine the range.

(a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/(x^2 + 1)$.

(b) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin(x)$. (Assume familiar facts about the sine function.)

(c) Define $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $f(n,m) = nm$.

(d) ... $f((x,y),(u$ ... e this

(f)it is one-to-one

range is {(a,b),a∈A}

Onto?

Case1: $|B| \geq 2$, not onto!

Case2: $|B| = 1$, onto

... $f((x,y),(u$ ... o you recognize this function?)

(f) Let A and B be nonempty and $b \in B$. Define $f : A \to A \times B$ by $f(a) = (a,b)$.

(g) Let X be a nonempty set. Define $f : \mathscr{P}(X) \to \mathscr{P}(X)$ by $f(A) = X \setminus A$.

(h) Let B be a fixed proper subset of a nonempty set X. We define a function $f : \mathscr{P}(X) \to \mathscr{P}(X)$ by $f(A) = A \cap B$.

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2 - x \text{ if } x < 1 \\ 1/x \quad \text{otherwise} \end{cases}.$$

Problem 14.13 Let $F([0,1])$ denote the set of all real-valued functions defined on the closed interval $[0,1]$. Define a new function $\phi : F([0,1]) \to \mathbb{R}$ by $\phi(f) = f(0)$. Is ϕ a function from $F([0,1])$ to $\mathbb{R}$? Is it one-to-one? Is it onto? Remember to prove all claims, and to provide examples where appropriate.

Solution *We note that ϕ is a function.*

This is not one-to-one because two different functions can map zero to the same value; that is, if we take f defined by $f(x) = 0$ for all x and we take g defined by $g(x) = x$ for all x, then $f \neq g$ but $\phi(f) = f(0) = 0 = g(0) = \phi(g)$.

This is onto, however. Take $x \in \mathbb{R}$ and note that the constant function f_x defined by $f_x(y) = x$ for all $y \in [0,1]$ is an element of $F([0,1])$. Therefore, $\phi(f_x) = f_x(0) = x$. So ϕ is onto.

Problem 15.11 Suppose that $f: A \to B$ and g_1 and g_2 are functions from B to A such that $f \circ g_1 = f \circ g_2$. Show that if f is bijective, then $g_1 = g_2$. If $g_1 \circ f = g_2 \circ f$ and f is bijective, must $g_1 = g_2$?

f∘g1=f∘g2,
so ∀x∈B, f(g1(x))=f(g2(x))
because f is bijection
g1(x)=g2(x) for all x∈B
so g1=g2

g1∘f=g2∘f,
so ∀x∈A, g1(f(x))=g2(f(x))
because f is bijection
so g1(x)=g2(x) for all x∈B
so g1=g2

If $g_1 \circ f = g_2 \circ f$, then likewise we can get $g_1 = g_2$ but $x \in dom(f^{-1}) = ran(f)$,

the domains of g_1 and g_2 may not be same. So g_1 and g_2 do not need to equal.

$\because f$ is bijection
$\therefore \forall x(x \in B \to \exists y(y \in A \land f(y) = x))$
又 $\because g_1 \circ f(y) = g_2 \circ f(y)$
$\therefore g_1(x) = g_2(x)$

Problem[#] **15.15** Let A, B, C, and D be nonempty sets. Let $f : A \to B$ and $g : C \to D$ be functions. Consider H defined on $A \cup C$ by

$$H(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in C \end{cases}.$$

consider A={1,2},B={3,4}C={1,2}D={5,6}
f(x)=x+2,g(x)=x+4
H(1)=f(1)=3 H(1)=g(1)=5
then H is not a function

consider A={1,2},B={3,4}C={3,4}D={5,6}
f(x)=x+2,g(x)=x+2
then H is a function
when $A \cap C = \emptyset$, H is a function

If $A \cap C \neq \emptyset$?

if $\forall x \in A \cap C, f(x) = g(x) \Rightarrow H(x)$ is a function

$\boldsymbol{\forall x \in A \cap C, f(x) = g(x) \Rightarrow H(x)}$ **is a function**

Problem 16.21 Suppose that $f : X \to Y$ is a function, and let B_1 and B_2 be subsets of Y.

(a) If $f^{-1}(B_1) = f^{-1}(B_2)$, must $B_1 = B_2$?

(b) Let f be a bijective function. Show that if $f^{-1}(B_1) = f^{-1}(B_2)$, then $B_1 = B_2$. Indicate clearly where you use one-to-one or onto. Did you use both?

- (b)

区分：$f^{-1}(a)$与$f^{-1}(\{a\})$

∀a∈B1

for $f^{-1}(B1)= f^{-1}(B2)$

∃b∈B2, $f^{-1}(a)=f^{-1}(b)$

because f(x) is onto

so there exist x0∈A,f(x0)=a,f(x0)=b

f is a function

so a=b

so a ∈B2

B1⊆B2

no, just use onto

$\because f$ is onto

$\because \forall b \in B_1$, 则存在$a \in A$, s.t.,

$$f(a) = b$$

$\therefore f(a) \in f^{-1}(B_1) = f^{-1}(B_2)$

又$\because f$ is a function

$\therefore f(a) = b \in B_2$

$\therefore B_1 \subseteq B_2$

同理可证：$B_2 \subseteq B_1$

$\therefore B_1 = B_2$

In what follows, $f : X \to Y$ will always denote a bijective function between two subsets, X and Y, of $\mathbb{R}$ satisfying $f^{-1} = 1/f$.

1. What can you say about the domain and range of such a function?
2. Find an example of such a function, where the domain of f consists of a single point. $f = \{(1,1)\}$
3. Find an example of such a function on a domain consisting of two points. $f = \{(1,1),(-1,-1)\}$
4. Can such a function f exist on the integers? Why or why not?
5. Show that $(f \circ f)(x) = 1/x$ and $f(1/f(x)) = x$ for all $x \in X$.
6. Show that $f(1/x) = 1/f(x)$ for all $x \in X$.
7. Define a function $g : (\mathbb{R} \setminus \{0\}) \to (\mathbb{R} \setminus \{0\})$ by

$$g(x) = \begin{cases} -x^3, & \text{if } x > 0 \\ -1/(x^{1/3}), & \text{if } x < 0. \end{cases}$$

5.

$$(f \circ f)(x) = f(f(x)) = \frac{1}{f^{-1}(f(x))} = \frac{1}{(f^{-1} \circ f)(x)} = \frac{1}{x}$$

$$f\left(\frac{1}{f(x)}\right) = f\left(f^{-1}(x)\right) = x$$

$$f^{-1} = \frac{1}{f} \Rightarrow f = \frac{1}{f^{-1}} \Rightarrow f \circ (f \circ f^{-1})(x) = \frac{1}{f^{-1}(x)} \Rightarrow (f \circ f)\left(f^{-1}(x)\right) = \frac{1}{f^{-1}(x)} \Rightarrow (f \circ f)(x) = \frac{1}{x}$$

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